# First order hyperbolic equations approximating Defocusing NLS equation

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# Introduction : Euler's equation for compressible fluids

A Lagrangian :

$$L = \int_{\Omega_t} \left( \frac{\rho \left|\mathbf{u}\right|^2}{2} - \rho e(\rho) \right) d\Omega_t$$

A differential constraint :

$$\rho_t + \mathrm{div}(\rho \mathbf{u}) = 0$$

$\Longrightarrow$ The corresponding Euler-Lagrange equation:

$$(\rho \mathbf{u})_t + \mathrm{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + p(\rho)\right) = \mathbf{0}; \quad p(\rho) = \rho^2 e'(\rho)$$

# Dispersive models in mechanics

1. Surface waves with surface tension [Nikolayev, Gavrilyuk, Gouin 2006] :

$$\mathcal{L}(\mathbf{u}, h, \nabla h) = \int_{\Omega_t} \left( \frac{h\,|\mathbf{u}|^2}{2} - \frac{gh^2}{2} - \sigma \frac{|\nabla h|^2}{2} \right) d\Omega_t$$

2. Shallow water equations described by Serre-Green-Naghdi equations [Salmon (1998)]:

$$\mathcal{L}(u, h, \dot{h}) = \int_{\Omega_t} \left( \frac{hu^2}{2} - \frac{gh^2}{2} + \frac{h\dot{h}^2}{6} \right) d\Omega_t$$

# Euler-Korteweg-Van Der Waals type systems

$$L = \int_{\Omega_t} \mathcal{L}(\mathbf{u}, \rho, \nabla\rho)\, d\Omega_t = \int_{\Omega_t} \left( \frac{\rho\,|\mathbf{u}|^2}{2} - \rho e(\rho) - K(\rho)\frac{|\nabla\rho|^2}{2} \right)\, d\Omega_t$$

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0 \\ \partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\rho) = \rho\nabla\left(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\right) \end{cases}$$

**$K(h) = \sigma$ : constant capillarity**

$$\partial_t(h\mathbf{u}) + \operatorname{div}(h\mathbf{u}\otimes\mathbf{u}) + \nabla p(h) = \sigma h\nabla\left(\Delta h\right)$$

**$K(\rho) = \frac{1}{4\rho}$ : Quantum capillarity / DNLS equation**

$$\partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u} + \frac{1}{4\rho}\nabla\rho\otimes\nabla\rho) + \nabla\left(\frac{\rho^2}{2} - \frac{1}{4}\Delta\rho\right) = 0$$

# Euler-Korteweg type systems

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \rho \nabla \left( K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2 \right) \end{cases}$$

- **Ph.D Objective** $\Rightarrow$ Make it first order hyperbolic !

**Hyperbolic equations** ✓

- Wave-like behaviour.
- perturbations propagate at finite speeds.
- Mathematically well-posed equations.

# The Non-Linear Schrödinger equation

$$i\epsilon\psi_t + \frac{\epsilon^2}{2}\Delta\psi - f\left(|\psi|^2\right)\psi = 0 \qquad ; \qquad \epsilon = \frac{\hbar}{m}$$

- A wide range of applications: Nonlinear optics, quantum fluids, surface gravity waves.
- The 1d-equation is completely integrable. [Zakharov,Manakov 1974]
- Construction of analytical solutions is possible.
- In what follows and for simplicity we take $\epsilon = 1$ and consider the cubic NLS equation $f\left(|\psi|^2\right) = |\psi|^2$

# Hydrodynamic NLS

$$\boxed{i\psi_t + \frac{1}{2}\Delta\psi - |\psi|^2\psi = 0}$$

## The Madelung transform (1927)

$$\psi(\mathbf{x},t) = \sqrt{\rho(\mathbf{x},t)}e^{i\theta(\mathbf{x},t)} \qquad \mathbf{u} = \nabla\theta$$

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0 \\ (\rho\mathbf{u})_t + \mathrm{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + \Pi\right) = 0 \end{cases}$$

with : $\quad \Pi = \left(\frac{\rho^2}{2} - \frac{1}{4}\Delta\rho\right)\mathbf{Id} + \frac{1}{4\rho}\nabla\rho\otimes\nabla\rho$

# A Lagrangian for DNLS equation

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0 \\ (\rho\mathbf{u})_t + \mathrm{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + \left(\frac{\rho^2}{2} - \frac{1}{4}\Delta\rho\right)\mathbf{Id} + \frac{1}{4\rho}\nabla\rho\otimes\nabla\rho\right) = 0 \end{cases}$$

$$\boxed{\mathcal{L}(\mathbf{u},\rho,\nabla\rho) = \int_{\Omega_t}\left(\rho\frac{|\mathbf{u}|^2}{2} - \frac{\rho^2}{2} - \frac{1}{4\rho}\frac{|\nabla\rho|^2}{2}\right)d\Omega_t}$$

$$E = \rho\frac{|\mathbf{u}|^2}{2} + \frac{\rho^2}{2} + \frac{1}{4\rho}\frac{|\nabla\rho|^2}{2}$$

**Energy conservation law:**

$$\frac{\partial E}{\partial t} + \mathrm{div}\left(E\mathbf{u} + \Pi\mathbf{u} - \frac{1}{4}\dot{\rho}\nabla\rho\right) = 0 \quad ; \quad \dot{\rho} = \rho_t + \mathbf{u}\cdot\nabla\rho$$

# Hyperbolic approximations

- Hyperbolic heat conduction equation [Cattaneo 1958].

$$u_t = u_{xx} \quad \Rightarrow \begin{cases} u_t = q_x \\ q_t = (u_x - q)/\tau \end{cases} \qquad \tau \ll 1$$

- Hyperbolic approximation of dispersive shallow water equations [Liapidevskii, Gavrilova 2008].
- Hyperbolic Navier-Stokes equations [Peshkov, Romenskii 2016]

# Augmented Lagrangian approach [Favrie,Gavrilyuk 2017]

## The objective

Obtain a new Lagrangian whose Euler-Lagrange equations :

- are hyperbolic.
- approximate NLS equations in a certain limit.

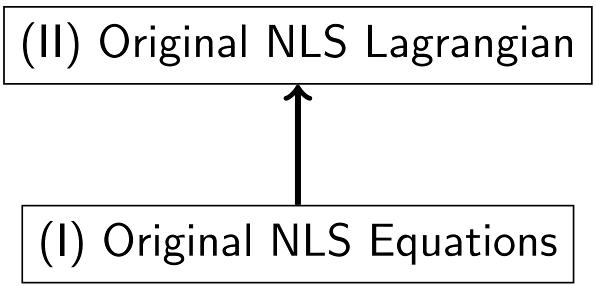
## Summary of key Ideas

- Consider a new variable that closely approximates $\rho$.
- Take its gradient as an independent variable.
- Rederive new system.

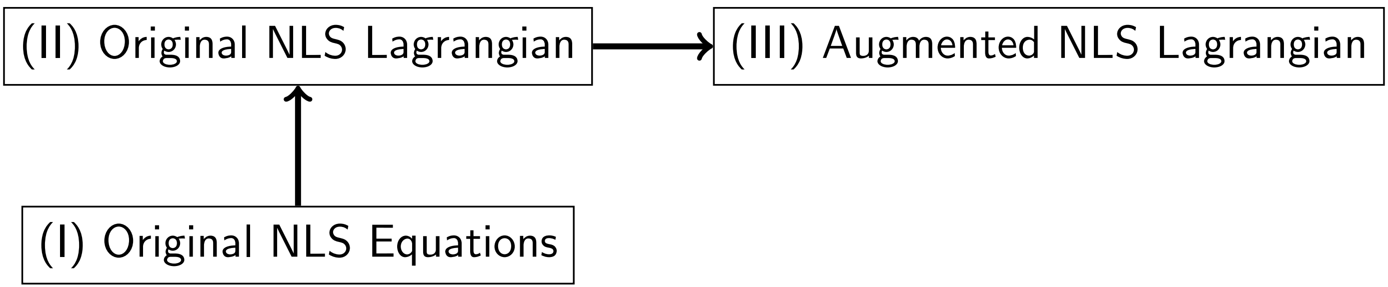
# Main Approach

(I) Original NLS Equations

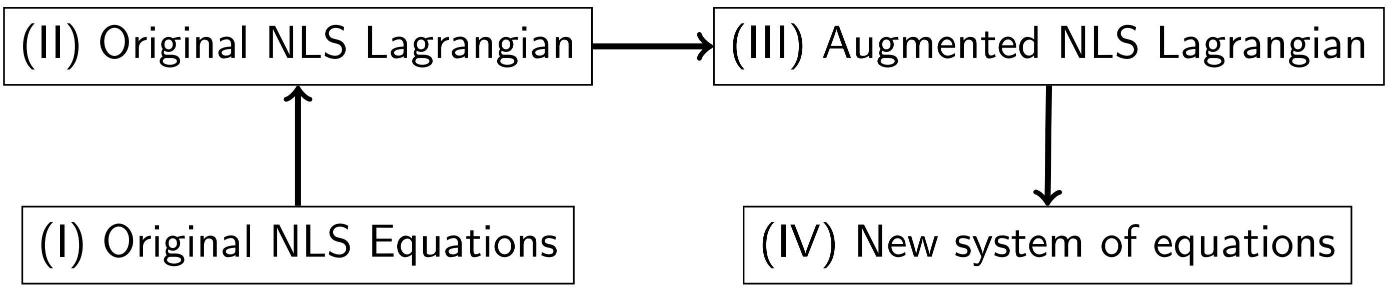
# Main Approach



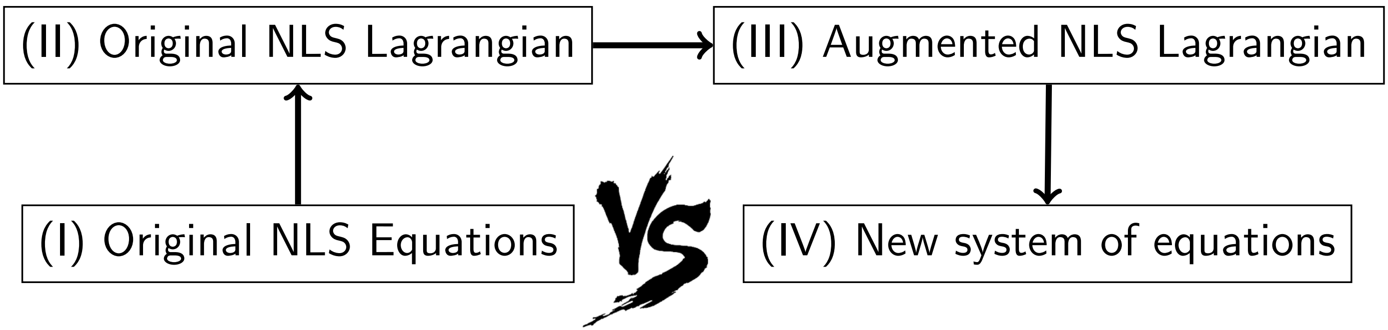
# Main Approach



# Main Approach



# Main Approach



# The concept : The relaxation part

$$\mathcal{L}(\mathbf{u}, \rho, \nabla\rho) = \int_{\Omega_t} \left( \frac{\rho\,|\mathbf{u}|^2}{2} - \rho e(\rho) - K(\rho)\frac{|\nabla\rho|^2}{2} \right) d\Omega_t$$

$$\partial_t \rho + \mathrm{div}(\rho\mathbf{u}) = 0$$

## 'Augmented' Lagrangian approach

$$\tilde{\mathcal{L}}(\mathbf{u}, \rho, \eta, \nabla\eta, \dot{\eta}) \qquad (\eta \longrightarrow \rho)$$

$$\tilde{\mathcal{L}} = \int_{\Omega_t} \left( \rho\frac{|\mathbf{u}|^2}{2} - \rho e(\rho) - K(\rho)\frac{|\nabla\eta|^2}{2} - \frac{1}{2\alpha}\rho\left(\frac{\eta}{\rho} - 1\right)^2 + \frac{\beta\rho}{2}\dot{\eta}^2 \right) d\Omega_t$$

$\frac{1}{2\alpha}\rho\left(\frac{\eta}{\rho} - 1\right)^2$ : Penalty

$\frac{\beta\rho}{2}\dot{\eta}^2$ : Regularization

# Types of variations

Two types of variations will be considered :

$$\tilde{\mathcal{L}}(\overbrace{\mathbf{u}, \rho, \dot{\eta}}^{I}, \underbrace{\dot{\eta}, \eta, \nabla\eta}_{II}) \qquad \dot{\eta} = \eta_t + \mathbf{u} \cdot \nabla\eta$$

- Type I : Virtual displacement of the continuum:

$$\hat{\delta}\rho = -\mathrm{div}(\rho\delta\mathbf{x}) \qquad \hat{\delta}\mathbf{u} = \dot{\delta\mathbf{x}} - \nabla\mathbf{u}\cdot\delta\mathbf{x} \qquad \hat{\delta}\dot{\eta} = \hat{\delta}\mathbf{u}\cdot\nabla\eta$$

- Type II : Variations with respect to $\eta$

$$\delta\nabla\eta = \nabla\delta\eta \qquad \hat{\delta}\dot{\eta} = (\delta\eta)_t + \mathbf{u}\cdot\nabla\delta\eta$$

# Augmented system Euler-Lagrange Equations

- Type I : Virtual displacement of the continuum:

$$\boxed{\frac{\partial \rho \mathbf{u}}{\partial t} + \mathrm{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + \Pi \mathbf{Id} + K(\rho)\nabla\eta \otimes \nabla\eta\right) = 0}$$

where:

$$\Pi = \left(\rho^2 e'(\rho) + \frac{1}{2}\left(\rho K'(\rho) - K(\rho)\right)|\nabla\eta|^2 + \frac{\eta}{\alpha}\left(1 - \frac{\eta}{\rho}\right)\right)$$

- Type II : Variations with respect to $\eta$:

$$\boxed{(\rho\dot{\eta})_t + \mathrm{div}\left(\rho\dot{\eta}\mathbf{u} - \frac{K(\rho)}{\beta}\nabla\eta\right) = \frac{1}{\alpha\beta}\left(1 - \frac{\eta}{\rho}\right)}$$

# Augmentation and closure of the system

Independent variables : $\mathbf{p} = \nabla\eta$ and $w = \dot{\eta}$.

**1. Definition of $w = \dot{\eta}$**

$$w = \dot{\eta} = \eta_t + \mathbf{u}\cdot\nabla\eta \quad \Longrightarrow \quad \boxed{(\rho\eta)_t + div(\rho\eta\mathbf{u}) = \rho w}$$

**2. Evolution of $\mathbf{p} = \nabla\eta$**

$$\begin{aligned} \nabla w &= \nabla(\eta_t + \mathbf{u}\cdot\nabla\eta) \\ &= (\nabla\eta)_t + \nabla(\mathbf{u}\cdot\nabla\eta) \\ \Longrightarrow \quad & (\nabla\eta)_t + \nabla(\mathbf{u}\cdot\nabla\eta - w) = 0 \\ \Longrightarrow \quad & \boxed{\mathbf{p}_t + \mathrm{div}((\mathbf{p}\cdot\mathbf{u} - w)\mathbf{Id}) = 0} \end{aligned}$$

**2'. Initial condition for p : $\mathbf{p}_{t=0} = (\nabla\eta)_{t=0}$**

# Augmented system for NLS equation

The augmented system reads as :

$$
\begin{cases}
\partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0 \\
\partial_t(\rho \mathbf{u}) + \mathrm{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + \mathcal{P}\right) = 0 \\
\partial_t(\rho w) + \mathrm{div}\left(\rho w \mathbf{u} - \frac{1}{4\rho\beta}\mathbf{p}\right) = \frac{1}{\alpha\beta}\left(1 - \frac{\eta}{\rho}\right) \\
\partial_t(\rho\eta) + \mathrm{div}(\rho\eta\mathbf{u}) = \rho w \\
\partial_t \mathbf{p} + \mathrm{div}\left((\mathbf{p}\cdot\mathbf{u} - w)\,\mathbf{Id}\right) = 0; \qquad \mathrm{curl}(\mathbf{p}) = 0
\end{cases}
$$

$$
\mathcal{P} = \left(\frac{\rho^2}{2} - \frac{1}{4\rho}|\mathbf{p}|^2 + \frac{\eta}{\alpha}(1 - \frac{\eta}{\rho})\right)\mathbf{Id} + \frac{1}{4\rho}\mathbf{p}\otimes\mathbf{p}
$$

- Does it approximate NLS ?
- Is it Hyperbolic ?
- Values of $\alpha$ and $\beta$ ?

# Face to face

## Original NLSE

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \operatorname{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + \Pi\right) = 0 \end{cases}$$

with : $$\Pi = \left(\frac{\rho^2}{2} - \frac{1}{4}\Delta\rho\right) \mathbf{Id} + \frac{1}{4\rho} \nabla\rho \otimes \nabla\rho$$

## Augmented system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + \mathcal{P}\right) = 0 \end{cases}$$

$$\mathcal{P} = \left(\frac{\rho^2}{2} - \frac{1}{4\rho}\left|\mathbf{p}\right|^2 + \frac{\eta}{\alpha}\left(1 - \frac{\eta}{\rho}\right)\right) \mathbf{Id} + \frac{1}{4\rho} \mathbf{p} \otimes \mathbf{p}$$

# Relaxation / Couplings

$$\underbrace{w_t + u w_x}_{\dot{w}} - \frac{1}{4\beta\rho^2} p_x + \frac{1}{4\beta\rho^3} p \rho_x = \frac{1}{\alpha\beta\rho}\left(1 - \frac{\eta}{\rho}\right)$$

$$\Rightarrow \rho - \eta = \alpha\beta\rho^2 \dot{w} - \frac{\alpha}{4} p_x + \frac{\alpha}{4\rho} p \rho_x$$

$$\Rightarrow \rho_x - \eta_x = \rho_x - p = \alpha\beta\left(\rho^2 \dot{w}\right)_x - \frac{\alpha}{4}\left(p_x - \frac{1}{\rho} p \rho_x\right)_x$$

$$\Rightarrow \frac{\eta}{\alpha}\left(1 - \frac{\eta}{\rho}\right) = \beta\rho\eta\dot{w} - \frac{\eta}{4\rho} p_x + \frac{\eta}{\rho^2} p \rho_x$$

$$= -\frac{1}{4}\rho_{xx} + \frac{1}{4\rho}\rho_x^2 + \mathcal{O}(\beta) + \mathcal{O}(\alpha)$$

# One-Dimensional case : Hyperbolicity

In order to study the hyperbolicity of this system, we write it in quasi-linear form :

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = \mathbf{q}$$

where:

$$\mathbf{U} = \left( \ \rho, u, w, p, \eta \ \right)^T \qquad \mathbf{q} = \left( \ 0, 0, \frac{1}{\alpha\beta\rho}\left(1 - \frac{\eta}{\rho}\right), 0, w \ \right)^T$$

$$\mathbf{A}(\mathbf{U}) = \begin{pmatrix} u & \rho & 0 & 0 & 0 \\ 1 + \frac{\eta^2}{\alpha\rho^3} & u & 0 & 0 & \frac{1}{\alpha\rho}\left(1 - \frac{2\eta}{\rho}\right) \\ \frac{p}{4\beta\rho^3} & 0 & u & -\frac{1}{4\beta\rho^2} & 0 \\ 0 & p & -1 & u & 0 \\ 0 & 0 & 0 & 0 & u \end{pmatrix}$$

# One-Dimensional case : Hyperbolicity

The eigenvalues $c$ of the matrix $\mathbf{A}$ are :

$$c = u, \ (c-u)^2_{\pm} = \frac{\left(\frac{1}{4\beta\rho^2} + \rho + \frac{\eta^2}{\alpha\rho^2}\right) \pm \sqrt{\left(-\frac{1}{4\beta\rho^2} + \rho + \frac{\eta^2}{\alpha\rho^2}\right)^2}}{2}.$$

The right-hand side is always positive. However, the roots can be multiple if

$$\frac{1}{4\beta\rho^2} = \rho + \frac{\eta^2}{\alpha\rho^2}.$$

In the case of multiple roots : We still get five linear independent eigenvectors. $\Longrightarrow$ the system is always hyperbolic

# Values of $\alpha$ and $\beta$

- Values have to be assigned : a criterion is needed.
- We can base this choice, for example, on the dispersion relation.

## Original DNLS dispersion relation

$$c_p^2 = \rho_0 + \frac{k^2}{4}$$

## Augmented DNLS dispersion relation

$$(c_p)^2 = \frac{\frac{1}{4\beta\rho_0^2} + \rho_0 + \frac{1}{\alpha} + \frac{1}{\alpha\beta\rho_0^2 k^2} - \sqrt{\left(\frac{1}{4\beta\rho_0^2} + \rho_0 + \frac{1}{\alpha} + \frac{1}{\alpha\beta\rho_0^2 k^2}\right)^2 - 4\left(\frac{1}{\alpha\beta\rho_0 k^2} + \frac{\rho_0 + \frac{1}{\alpha}}{4\beta\rho_0^2}\right)}}{2}$$

# Example estimation

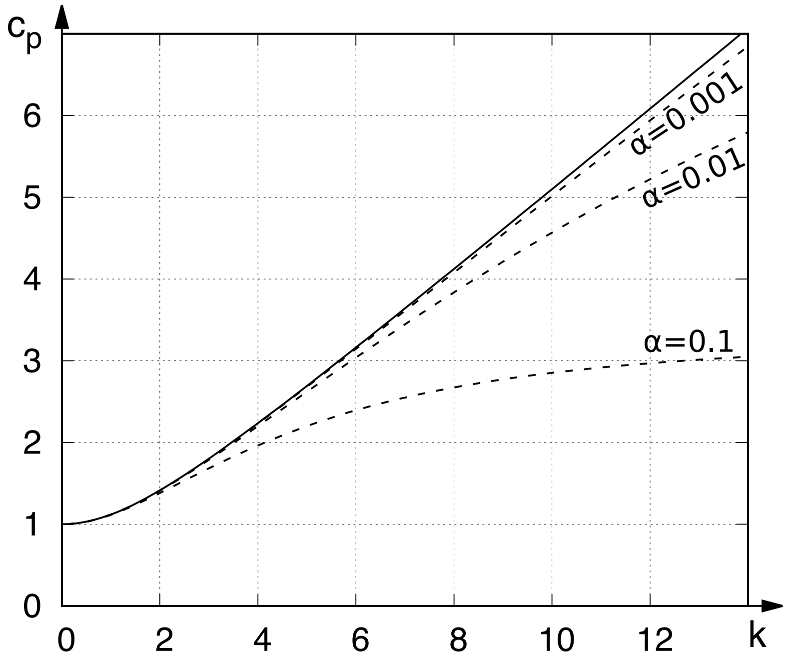



Figure 1: The dispersion relation $c_p = f(k)$ for the original model (continuous line) and the dispersion relation for the Augmented model (dashed lines) for different values of $\alpha$ and for $\beta = 10^{-4}$

# To summarize

1. Start from Euler-Korteweg equations.
2. Shift back to the Lagrangian.
3. Modify the Lagrangian (Relaxation, Augmentation).
4. Rederive the Euler-Lagrange equations + closure equations.
5. Write the scheme and do simulations.

These steps are reunited within a Mathematica code :

- Input : Total energy or Lagrangian.
- Output : ALL the Fortran lines needed for the code including fluxes, eigenvalues, source terms, etc

# Numerical scheme: IMEX-Type

**1-d system of equations to solve :**

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S}(\mathbf{U})$$

The idea is to solve the hyperbolic part explicitly and the source term implicitly in time according to the scheme :

$$\mathbf{U}^{\star} = \mathbf{U}^{n} - \gamma \frac{\Delta t}{\Delta x}\left(F^{n}_{i+\frac{1}{2}} - F^{n}_{i-\frac{1}{2}}\right) + \gamma \Delta t \mathbf{S}(\mathbf{U}^{\star})$$

$$\mathbf{U}^{n+1} = \mathbf{U}^{n} - (\gamma - 1)\frac{\Delta t}{\Delta x}\left(F^{n}_{i+\frac{1}{2}} - F^{n}_{i-\frac{1}{2}}\right) - (2-\gamma)\frac{\Delta t}{\Delta x}\left(F^{\star}_{i+\frac{1}{2}} - F^{\star}_{i-\frac{1}{2}}\right) + (1-\gamma)\Delta t S(\mathbf{U}^{\star}) + \gamma \Delta t S(\mathbf{U}^{n+1})$$

# Numerical scheme : Riemann solver

**Riemann Solver:** Rusanov.

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}\left(\mathbf{F}(\mathbf{U}^n_{i+1}) + \mathbf{F}(\mathbf{U}^n_i)\right) - \frac{1}{2}\kappa^n_{i+\frac{1}{2}}\left(\mathbf{U}^n_{i+1} - \mathbf{U}^n_i\right)$$

where $\kappa^n_{i+\frac{1}{2}}$ is obtained by using the Davis approximation :

$$\kappa^n_{i+1/2} = \max_j(|c_j(\mathbf{U}^n_i)|, |c_j(\mathbf{U}^n_{i+1})|),$$

where $c_j$ are the eigenvalues of the augmented system.

# Travelling wave solutions

- NLS equation equation admits travelling wave solutions :

$$\begin{cases} \rho(x,t) = b_1 - (b_1 - b_3)\mathrm{dn}^2\left(\sqrt{b_1 - b_3}\,(x - Ut)\,, s\right) \\ (b_1 > b_2 > b_3) \end{cases}$$

with $s$ the elliptic modulus satisfying the relation :

$$s^2 = \frac{b_2 - b_3}{b_1 - b_3}, \quad 0 < s < 1.$$

- For each fixed value of $0 < s < 1$, this solution is a periodic wave of amplitude $a$ and wavenumber $k$ given by :

$$a = \frac{b_2 - b_3}{2}, \quad k = \frac{\pi}{K(s)}\sqrt{\frac{2a}{s^2}}.$$

# Grey Solitons

obtained from previous solution in the limit $s \to 1$ :

$$\rho(x,t) = b_1 - \frac{b_1 - b_3}{\cosh^2\left(\sqrt{b_1 - b_3}\,(x - Ut)\right)} \qquad u(x,t) = U - \frac{b_1\sqrt{b_3}}{\rho(x,t)}$$

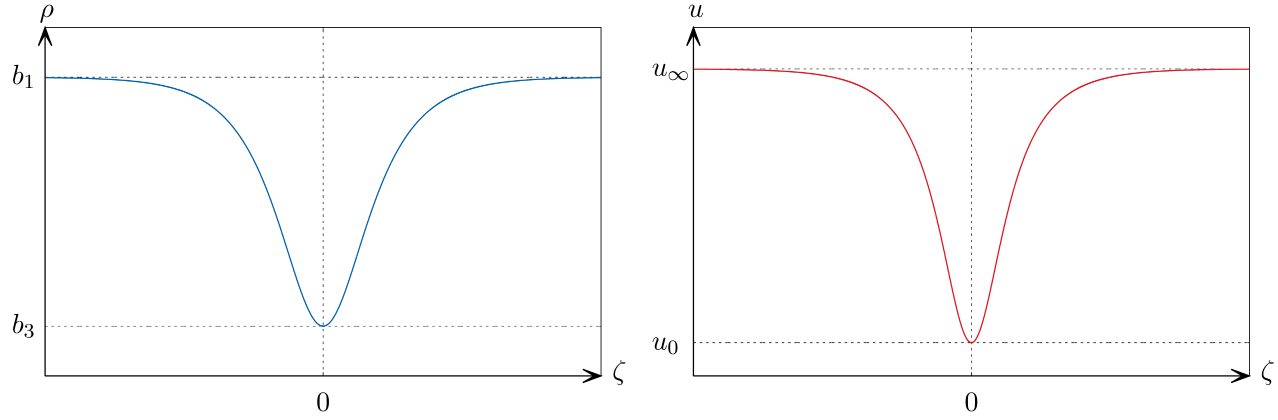



Figure 2: Grey soliton solution, for arbitrary values of the parameters $b_1$ and $b_3$ at $t = 0$

# Numerical solution for a grey soliton

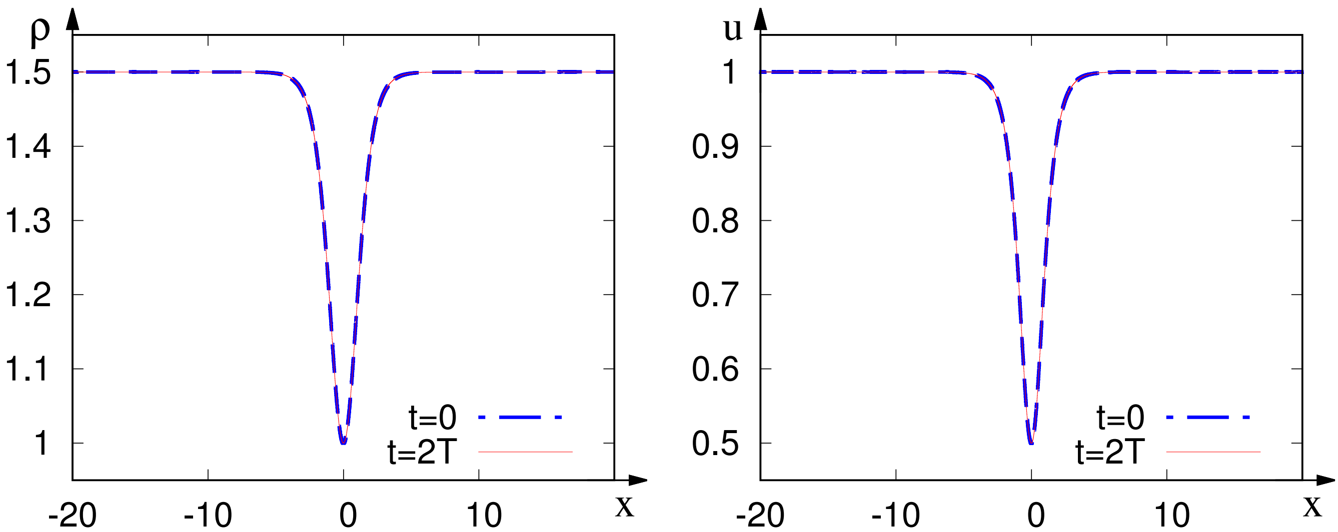



Figure 3: Numerical profiles of $\rho$ (left) and $u$ (right) for the grey soliton at $t = 0$ (dot-dashed line) and at $t = 2T$ (continuous line). The used domain is $L = [-20, 20]$ with $\Delta x = 0.0002$. Parameters used for the simulation are $b_1 = 1.5$, $b_3 = 1$, $U = 2$, $\beta = 10^{-4}$, $\alpha = 0.002$.

# A brief introduction to DSWs

Riemann problem in dispersionless hydrodynamics governed by Euler Equations :

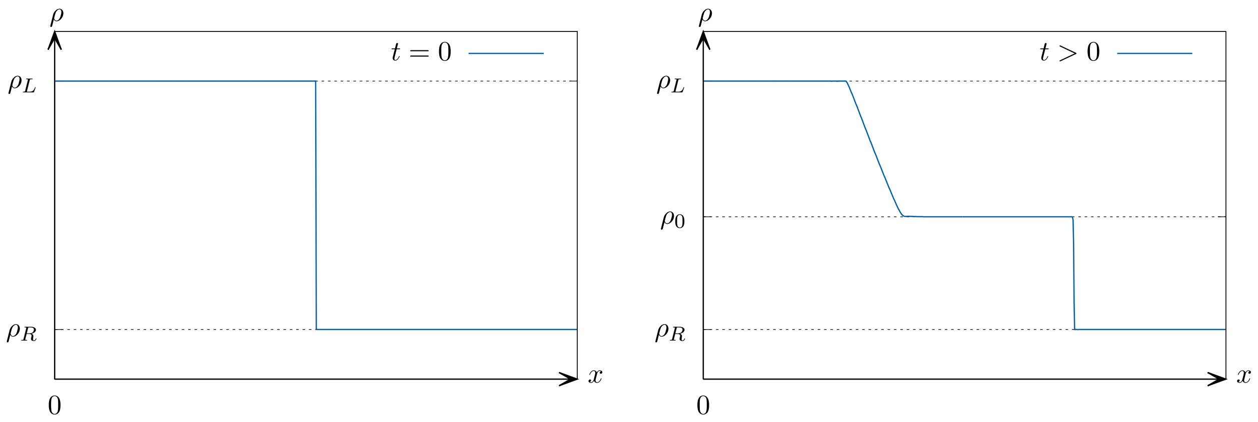



Figure 4: Shockwave solution to a Riemann problem for Euler Equations.

# A brief introduction to DSWs

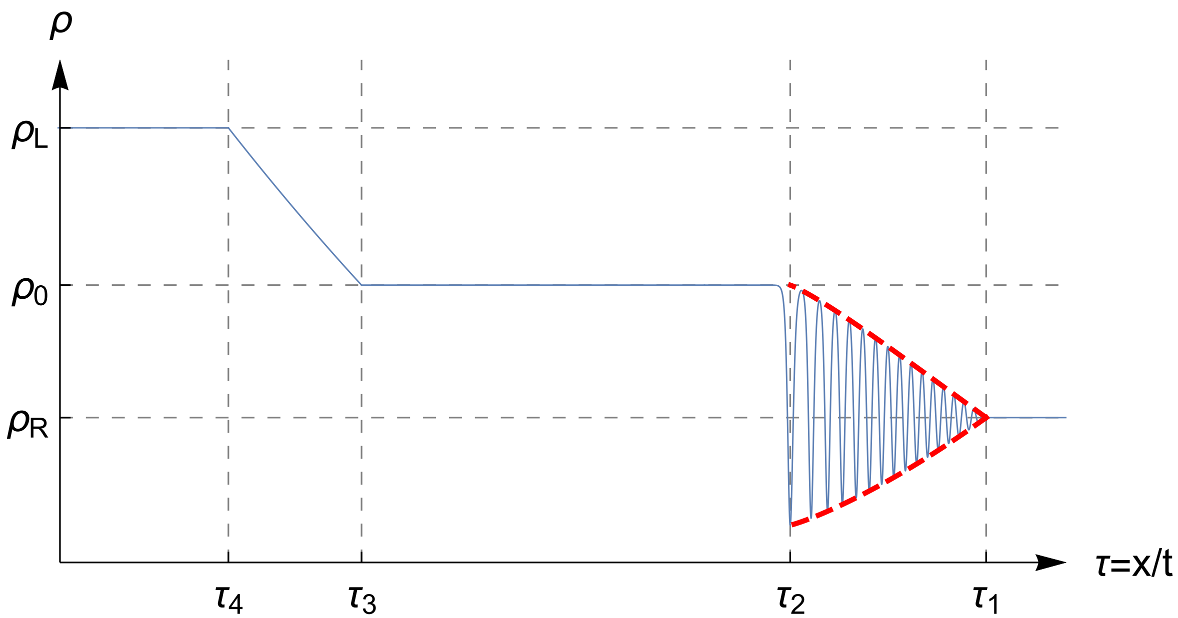



Figure 5: Asymptotic profile of the solution to NLS equation (continuous line) for the Riemann problem $\rho_L = 2$, $\rho_R = 1$ , $u_L = u_R = 0$. Oscillations shown at t=70

# A brief introduction to DSWs

Non exhaustive Literature :

- Writing Whitham equations for NLSE [Pavlov, 1987]
- Structure of dispersive shockwave [Gurevich, Krylov, 1987]
- Classification of DSWs arising from initial an discontinuity for NLSE [El et al. 1995]
- [Hoefer et al. 2008, ], [El, Hoefer 2016] ,... etc

# DSW Numerical results : $\rho$

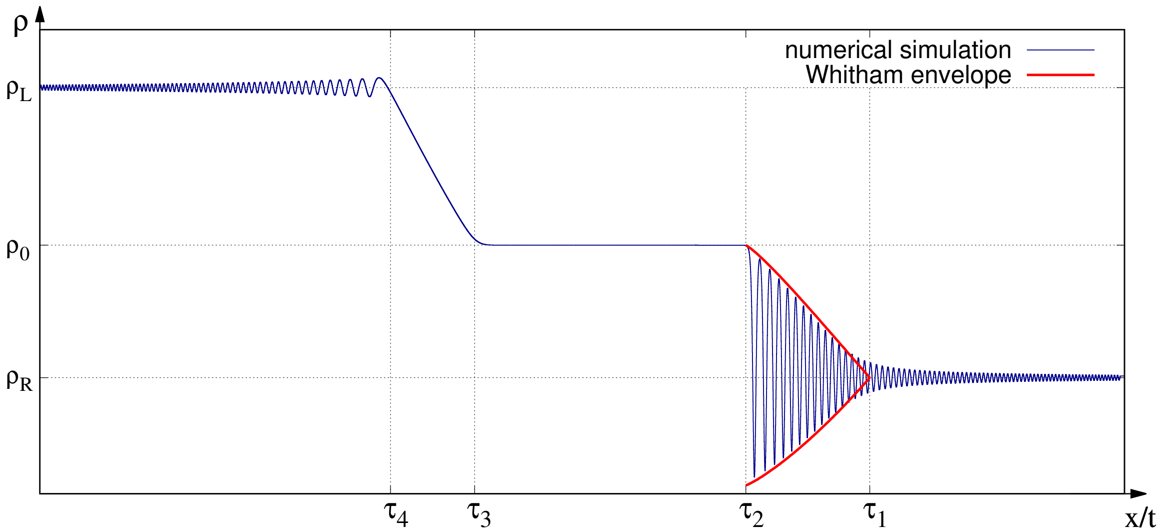



Figure 6: Comparison of the numerical result $\rho(x,t) = f(x/t)$ (blue line) with the asymptotic profile of the oscillations from Whitham's theory of modulations. t=70

# DSW Numerical results : $u$

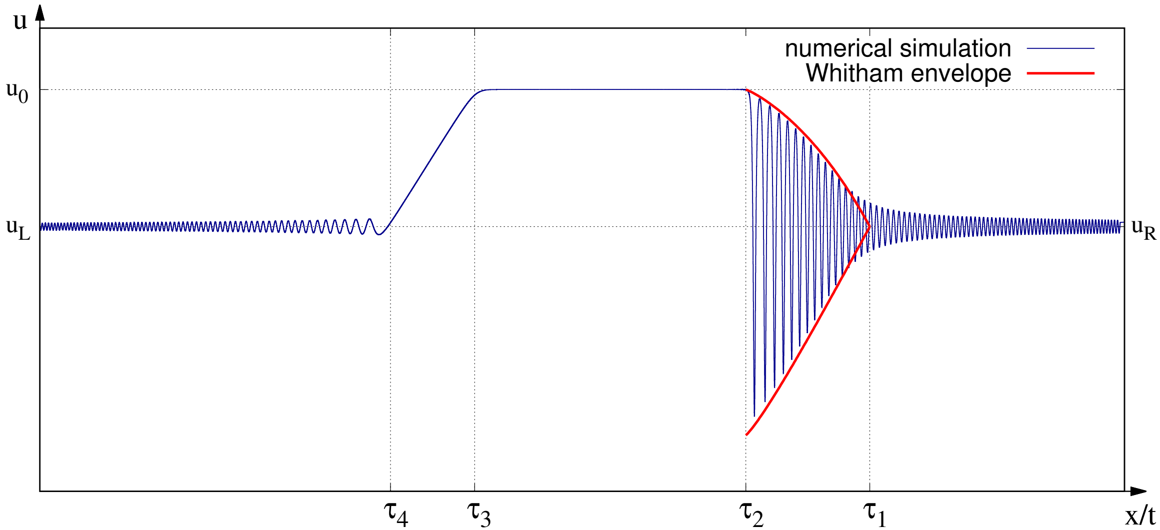



Figure 7: Comparison of the numerical result $u(x, t) = f(x/t)$ (blue line) with the asymptotic profile of the oscillations from Whitham's theory of modulations. t=70

# vanishing oscillations at the left constant state

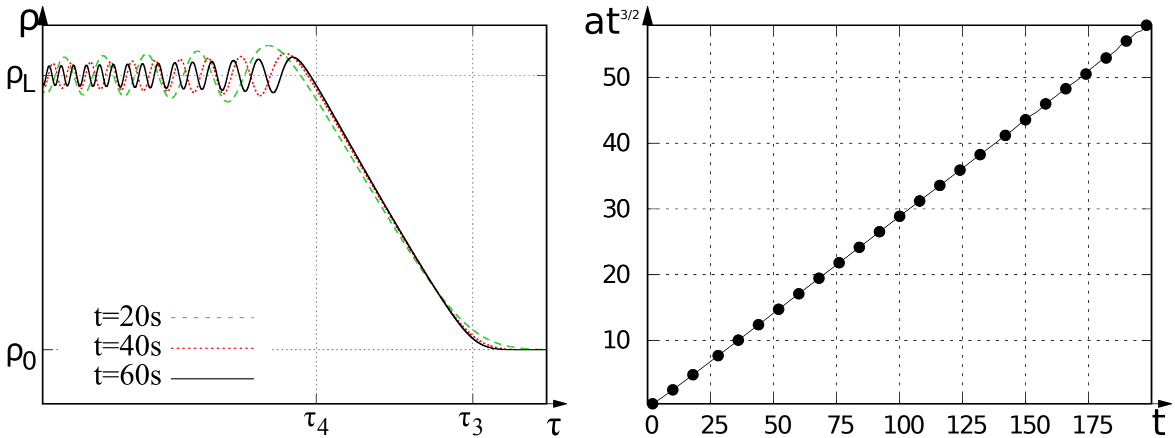



Figure 8: Vanishing oscillations at the vicinity of $\tau = \tau_4$. amplitude decreases as $\propto t^{-1/2}$.

# Equations for thin films flow

The same approach was applied for thin films flows with capillarity, which also are governed by an Euler-Korteweg type system :

$$h_t + (hu)_x = 0$$

$$(hu)_t + \left(hu^2 + \frac{gh^2}{2}\cos\theta + \frac{g^2 \sin(\theta)^2}{\nu^2} h^5 + \frac{\sigma}{2\rho} h_x^2 - \frac{\sigma}{\rho} h h_{xx}\right)_x = gh\sin(\theta) - \nu \frac{u}{h}$$

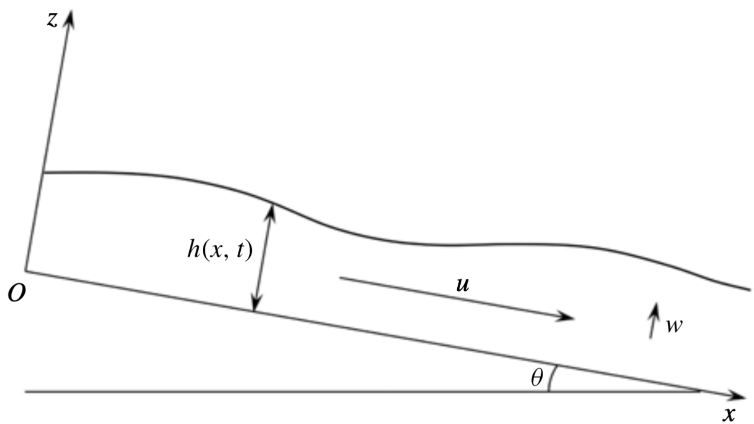



Figure 9: Sketch of the setting

# Liu & Gollub's experiment (1994)

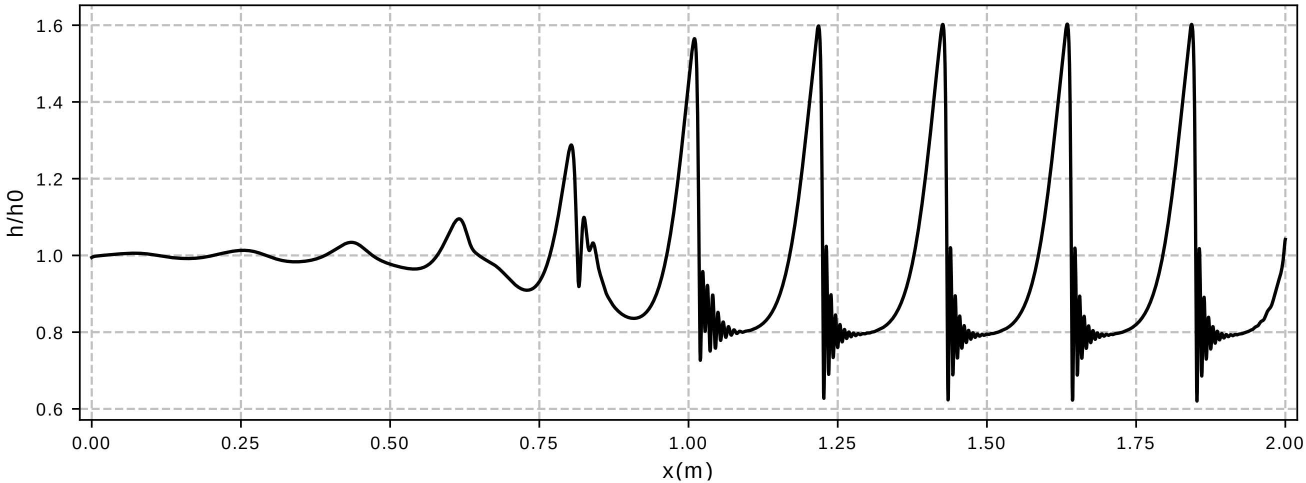



Figure 10: The Liu-Gollub experiment. The curve is the dimensionless depth of the wave profile in a 2.0 meter long canal, obtained with a forcing frequency f = 1.5Hz, imposed at the left boundary.

# Numerical result : f=1.5Hz

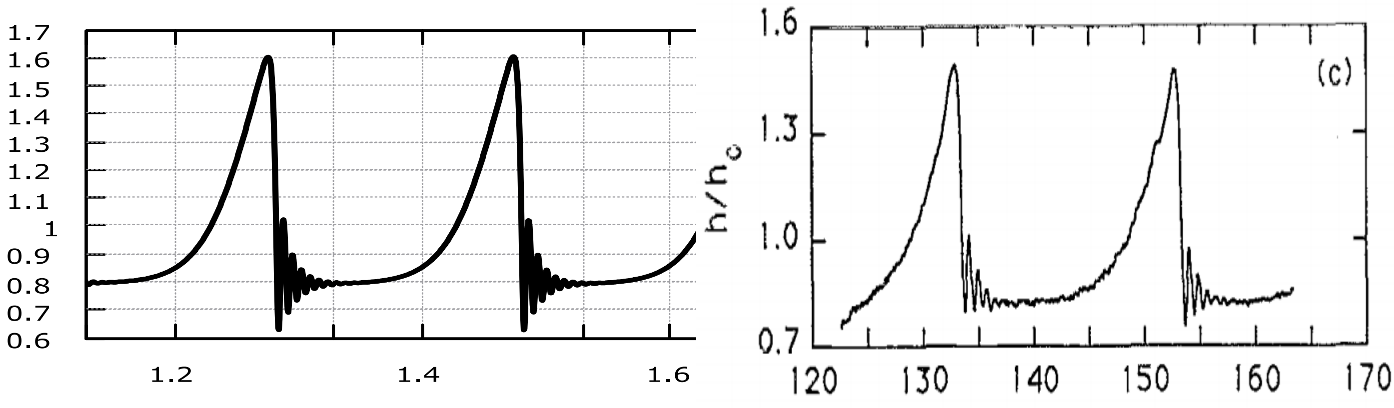



Figure 11: Comparison of the numerical simulation of the Liu-Gollub experiment with experimental data for f=1.5Hz ($\alpha = 0.005$, $\beta = 0.00003$, $\varepsilon = 0.0067$, nx=4000 ) boundary.

# Numerical result : f=3.0Hz

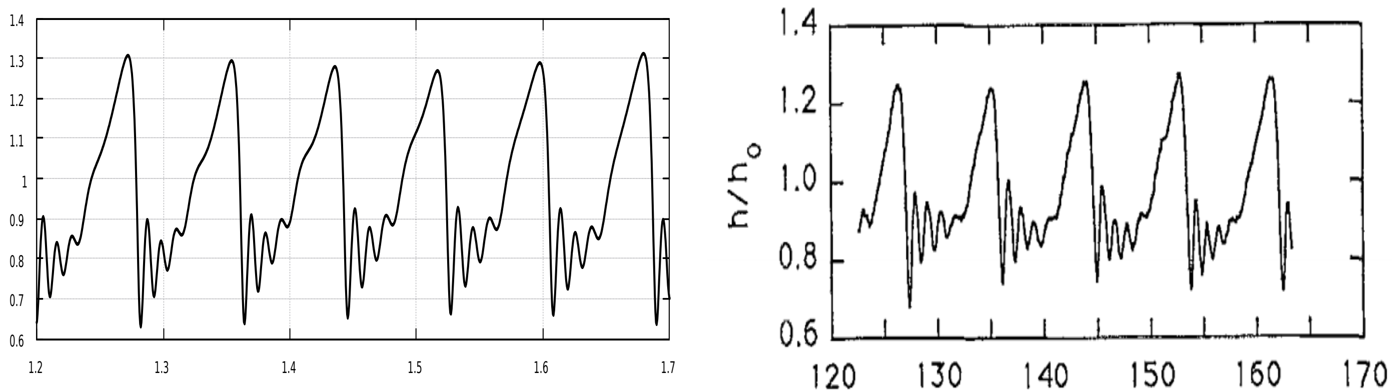



Figure 12: Comparison of the numerical simulation of the Liu-Gollub experiment with experimental data for f=3.0Hz ($\alpha = 0.005$, $\beta = 0.00003$, $\varepsilon = 0.0067$, nx=4000 ) boundary.

# Conclusion

- A first order hyperbolic approximation of defocusing NLS equations is presented.
- In 1-d, the system is unconditionally hyperbolic
- Asymptotic and numerical comparisons between the original and augmented system were done for both stationary and non stationary solutions.
- The approach is extendable to multidimensional case, provided suitable modifications are made.

# Perspectives

- limitations when $\rho \to 0$
- Application to other E-K systems
- Better numerics (boundary conditions, higher order, better performance, ...)
- Proper extension to multi-D.
- Applications to systems with non convex energies.

# Thank you for your attention

Full details in :

Firas Dhaouadi, Nicolas Favrie, and Sergey Gavrilyuk.
Extended Lagrangian approach for the defocusing nonlinear Schrödinger equation.
*Studies in Applied Mathematics*, 142(3):336–358, 2019.