Extended Lagrangian approach for the defocusing nonlinear Schrödinger equation

Firas Dhaouadi[1], Nicolas Favrie[2] and Sergey Gavrilyuk[2]

[1] : Université Paul Sabatier, Institut de Mathématiques de Toulouse, Toulouse Cedex 9, France.

[2] : Aix-Marseille Univ, CNRS, IUSTI, Marseille Cedex 13, France.

Correspondence to dhaouadi@insa-toulouse.fr

Abstract

We present an approximate first order hyperbolic model for the hydrodynamic form of the defocusing nonlinear Schrödinger equation (NLS). This Euler-Korteweg type system can be seen as an Euler-Lagrange equations to a Lagrangian submitted to a mass conservation constraint. Due to the presence of dispersive terms, such a Lagrangian depends explicitly on the gradient of density. The idea is to create a new dummy variable which accurately approximates the density via a penalty method. Then, we take its gradient as a new independent variable and apply Hamilton's principle. After adding suitable closure equations, the resulting system is a first order hyperbolic set of equations with stiff source terms and fast characteristic speeds. It is solved numerically using Godunov-type methods. Comparisons with an exact and asymptotic solutions to the one-dimensional cubic NLS is presented.

1. Defocusing NLS Equation

The defocusing cubic NLS equation has the following form:

$$i\psi_t + \frac{1}{2}\Delta\psi - |\psi|^2\,\psi = 0.$$

The change of variables $\psi = \sqrt{\rho}\, e^{i\theta}$, $\nabla\theta = \mathbf{u}$, known as Madelung's Transform permits to cast the previous equation into hydrodynamic form :

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \operatorname{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + \mathbf{\Pi}\right) = 0, \end{cases}$$

with $\mathbf{\Pi} = \left(\frac{\rho^2}{2} - \frac{1}{4}\Delta\rho\right)\mathbf{Id} + \frac{1}{4\rho}\nabla\rho\otimes\nabla\rho.$

This particular case of Euler-Korteweg type systems can be seen as the Euler-Lagrange equations for the Lagrangian :

$$\mathcal{L}(\rho,\mathbf{u},\nabla\rho) = \int_{\Omega_t}\left(\rho\frac{|\mathbf{u}|^2}{2} - \frac{\rho^2}{2} - \frac{1}{4\rho}\frac{|\nabla\rho|^2}{2}\right)d\Omega$$

2.Extended Lagrangian formulation

Let us consider a new variable η. The idea is to substitute $\nabla\rho$ by $\nabla\eta$ and to guarantee the convergence of η to ρ in a certain limit. To do that, let us consider the extended Lagrangian :

$$\mathcal{L}^e(\rho,\mathbf{u},\eta,\nabla\eta,\dot\eta) = \int_{\Omega_t}\left(\rho\frac{|\mathbf{u}|^2}{2} - \frac{\rho^2}{2} - \frac{1}{4\rho}\frac{|\nabla\eta|^2}{2} - \frac{\lambda}{2}\rho\left(\frac{\eta}{\rho}-1\right)^2 + \frac{\beta}{2}\rho\dot\eta^2\right)d\Omega,$$

where $\lambda \gg 1$ and $\beta \ll 1$. The term $\frac{\lambda}{2}\rho\left(\frac{\eta}{\rho}-1\right)^2$ is a classical penalty term. When $\lambda\to\infty$, the difference $(\eta/\rho - 1)$ vanishes. The term $\frac{\beta}{2}\rho\dot\eta^2$ is necessary in order to regularize the time evolution of η and to ensure the hyperbolicity of the new governing equations. We denote $\mathbf{p} = \nabla\eta$ and $w = \dot\eta$. Using a variational principle to the Lagrangian $\mathcal{L}^e$ under the mass conservation constraint, one obtains the equations :

$$\begin{aligned}
&\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = 0,\\
&\frac{\partial\rho\mathbf{u}}{\partial t} + \operatorname{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + \left(\frac{\rho^2}{2} - \frac{1}{4\rho}|\mathbf{p}|^2 + \eta\lambda\left(1-\frac{\eta}{\rho}\right)\right)\mathbf{Id} + \frac{1}{4\rho}\mathbf{p}\otimes\mathbf{p}\right) = 0,\\
&\frac{\partial\rho\eta}{\partial t} + \operatorname{div}(\rho\eta\mathbf{u}) = \rho w,\\
&\frac{\partial\rho w}{\partial t} + \operatorname{div}\left(\rho w\mathbf{u} - \frac{1}{4\rho\beta}\mathbf{p}\right) = \frac{\lambda}{\beta}\left(1-\frac{\eta}{\rho}\right),\\
&\frac{\partial\mathbf{p}}{\partial t} + \operatorname{div}\left((\mathbf{p}\cdot\mathbf{u} - w)\,\mathbf{Id}\right) = 0; \quad \operatorname{curl}(\mathbf{p}) = 0.
\end{aligned}$$

In the one-dimensionnal case, this system is hyperbolic and the characteristic speeds c are given by:

$$c = u,\ (c-u)^2_{\pm} = \frac{1}{2}\left(\frac{1}{4\beta\rho^2} + \rho + \frac{\lambda\eta^2}{\rho^2} \pm \left|-\frac{1}{4\beta\rho^2} + \rho + \frac{\lambda\eta^2}{\rho^2}\right|\right).$$

3.Dispersion relation comparison

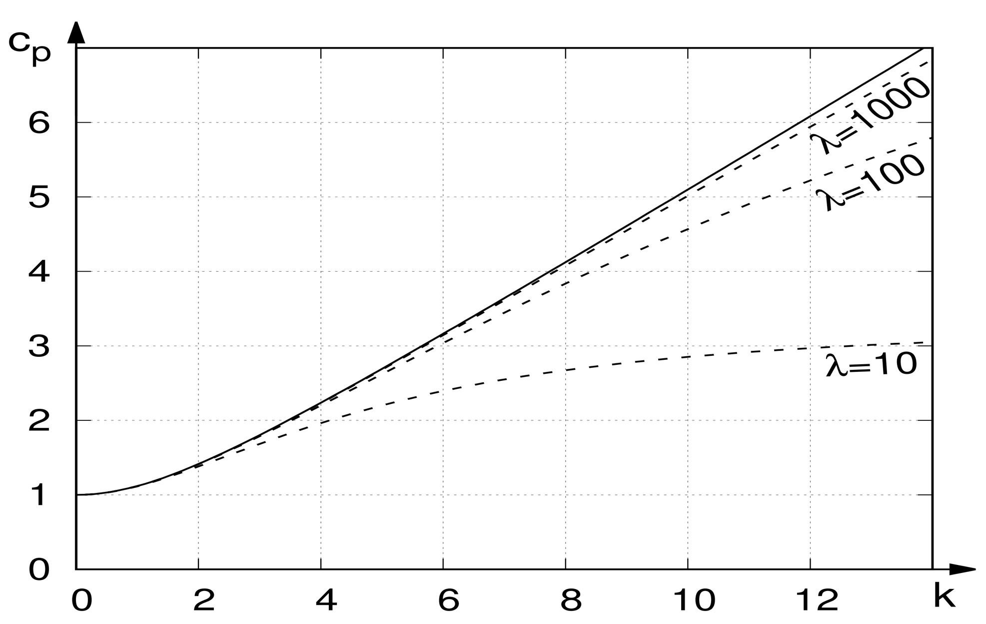

Figure 1: The dispersion relation, for the hydrodynamic NLS equation (continuous line) and for the extended Lagrangian (dashed lines) for $\beta = 10^{-4}$ and different values of λ.

4.Numerical Resolution

The shown results are obtained by the MUSCL-Hancock extension to the Godunov scheme, using Rusanov solver with a MIN-MOD limiter.In the 1-D case, the system can be written as :

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{S}(\mathbf{U}),$$

where $\mathbf{U}$, $\mathbf{F}$ and $\mathbf{S}$ are respectively the vector of conservative variables, flux vector and source term. A splitting strategy is applied. Hence, at each time step, the numerical resolution is split into a hyperbolic part :

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0,$$

and an ordinary differential equation part :

$$\mathbf{U}_t = \mathbf{S}(\mathbf{U}).$$

5. Results

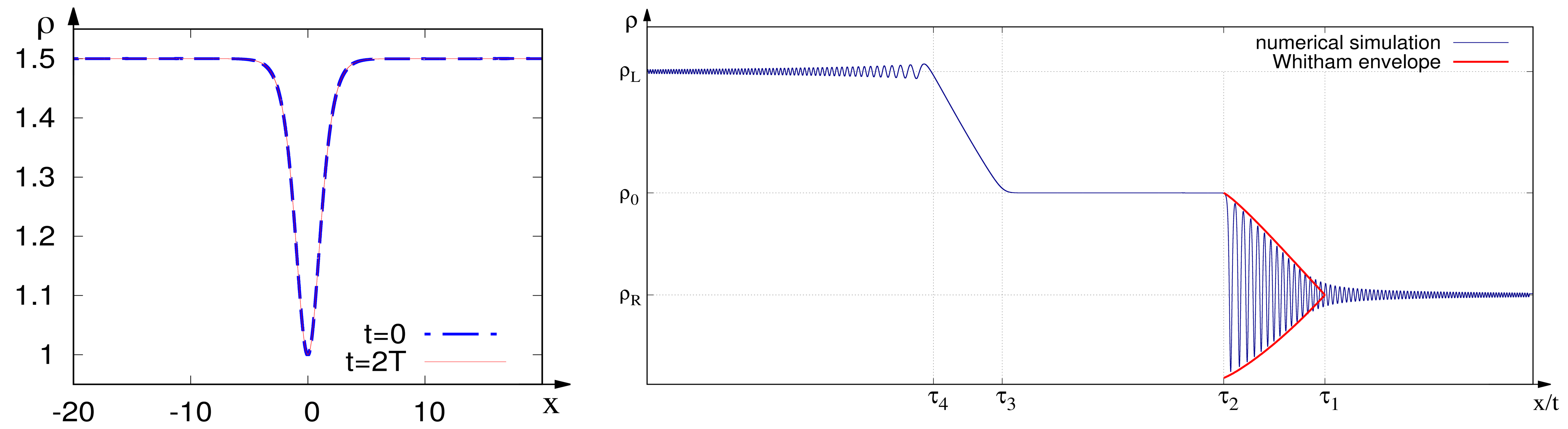

Figure 2: Left: Numerical results for a gray soliton with periodic BC ($\Delta x = 2.10^{-4}, \lambda = 500, \beta = 10^{-4}$). Right: Comparison of a simulated dispersive shock with its asymptotic envelope obtained through Whitham's Theory of modulations ($\Delta x = 6.66\ 10^{-4}, \lambda = 300, \beta = 2.10^{-5}$).

6. Conclusions

Conclusions :

1. The hydrodynamic form of the one-dimensional cubic NLS equation is solved by an extended Lagrangian method.
2. The obtained system of equations is first order hyperbolic with stiff source terms and fast characteristic speeds.
3. the numerical results show good agreement in the stationary and non-stationary case.

7. Reference

[1] Firas Dhaouadi, Nicolas Favrie, and Sergey Gavrilyuk. Extended lagrangian approach for the defocusing nonlinear schrödinger equation. *Studies in Applied Mathematics*, 142(3):336–358, 2019.