Hyperbolic formulations of dispersive equations in continuum mechanics

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Joint work with
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Surface tension / capillarity

- Euler-Korteweg equations : Fluid flow + Surface tension.
- Surface tension = Tendency of a fluid to shrink and minimize its surface.
- Examples in nature : Droplet shape, ripples on the water surface, water striders, etc...



Photos credits : pexels.com

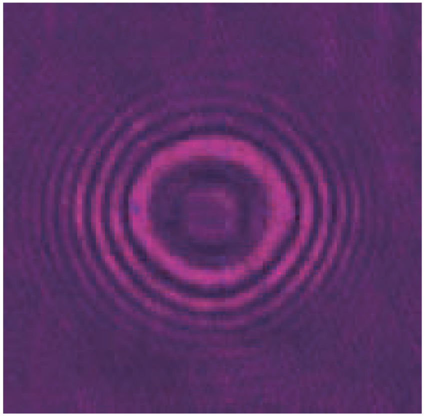
Industrial applications

- Hydrophobic sprays (clothes, shoes, car glass, buildings, etc)
- Anti-icing liquids for plane wings, heating systems,...
- Nuclear evaporators, pharmaceutical applications...

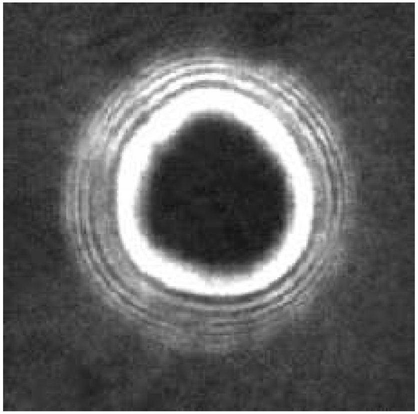
Photo credits : Ave Calvar Martinez pexels.com

Colin cutler boldmethod.com

More theoretical examples



(a)
Dispersive shock in nonlinear optics



(b)
Blast wave in a Bose-Einstein condensate

(c)
Capillary ripples on the water surface

Photo credits (a) : Wan, W. et. al. Dispersive superfluid-like shock waves in nonlinear optics. Nature Phys 3, 46–51 (2007).
(b) : Hoefer, M. A et. al (2006). Dispersive and classical shock waves in Bose-Einstein condensates and gas dynamics. Physical Review A, 74(2).
(c) : Wikipedia, picture taken by Roger McLassus (Creative commons)

Euler-Korteweg equations

The equations write :

$$\begin{cases} \rho_t + \mathrm{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \rho \nabla \left(K(\rho) \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2 \right) \end{cases}$$

where $\rho = \rho(\mathbf{x}, t)$, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and $(\mathbf{x}, t) \in \mathbb{R}^d \times [0, T]$

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- $K(\rho) = \frac{1}{4\rho}$: **Quantum hydrodynamics**

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \operatorname{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + \frac{1}{4\rho}\nabla\rho \otimes \nabla\rho\right) + \nabla\left(\frac{\rho^2}{2} - \frac{1}{4}\Delta\rho\right) = 0 \end{cases}$$

Main objective

Given the Euler-Korteweg system of equations :

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Ultimate Motivation

Can we obtain a first-order hyperbolic reformulation of this model ?

More than just hyperbolic

We want a new model that:

- approximates Euler-Korteweg in some limit.
- is derived from a variational principle.
- is in line with the laws of thermodynamics.
- can be solved numerically with accurate numerical methods.

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Hyperbolic equations

- Finite speed propagation (in line with relativity).
- Mathematically well-posed equations.
- A very rich literature on numerical methods.

Outline

The Non-Linear Schrödinger equation

Expressed in terms of the complex scalar field $\psi(\mathbf{x}, t)$:

$$i\psi_t + \frac{1}{2}\Delta\psi - f(|\psi|^2)\psi = 0$$

- It has a wide range of applications:
  - Nonlinear optics
  - Quantum fluids
  - Surface gravity waves
- It is integrable in the 1-d case [Zakharov, Shabat 1972]

  $\Rightarrow$ Obtaining analytical solutions is possible.

Defocusing NLS equation

Particular case of a cubic non-linearity $f(|\psi|^2) = |\psi|^2$:

$$i\psi_t + \frac{1}{2}\Delta\psi - |\psi|^2\,\psi = 0$$

The Madelung transform (1927)

$$\psi(\mathbf{x},t) = \sqrt{\rho(\mathbf{x},t)}e^{i\theta(\mathbf{x},t)} \qquad \mathbf{u} = \nabla\theta$$

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0 \\ (\rho\mathbf{u})_t + \operatorname{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + \left(\frac{\rho^2}{2} - \frac{1}{4}\Delta\rho\right)\mathbf{I_d} + \frac{1}{4\rho}\nabla\rho\otimes\nabla\rho\right) = 0 \end{cases}$$

$\Rightarrow$ Corresponds to the Euler-Korteweg quantum hydrodynamic system in the case of a potential flow (irrotational velocity field).

Lagrangian for Quantum hydrodynamics system

The hydrodynamic form of NLS equation admits the following Lagrangian:

$$\mathcal{L} = \int_{\Omega_t} \left(\frac{\rho\,|\mathbf{u}|^2}{2} - \frac{\rho^2}{2} - \frac{1}{4\rho}\frac{|\nabla\rho|^2}{2} \right) \, d\Omega$$

Hamilton's principle : $a = \displaystyle\int_{t_0}^{t_1} \mathcal{L}\, dt$

$+$

Differential constraint : $\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0$

$$(\rho\mathbf{u})_t + \operatorname{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + \frac{1}{4\rho}\nabla\rho\otimes\nabla\rho\right) + \nabla\left(\frac{\rho^2}{2} - \frac{1}{4}\Delta\rho\right) = 0$$

Augmented Lagrangian - Attempt 1

Original Lagrangian

$$\mathcal{L}(\mathbf{u}, \rho, \nabla\rho) = \int_{\Omega_t} \left(\rho\frac{|\mathbf{u}|^2}{2} - \frac{\rho^2}{2} - \frac{1}{4\rho}\frac{|\nabla\rho|^2}{2} \right) d\Omega$$

$$\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0$$

'Augmented' Lagrangian approach

$$\tilde{\mathcal{L}}(\mathbf{u}, \rho, \eta, \nabla\eta) \qquad (\eta \longrightarrow \rho)$$

$$\tilde{\mathcal{L}} = \int_{\Omega_t} \left(\rho\frac{|\mathbf{u}|^2}{2} - \frac{\rho^2}{2} - \frac{1}{4\rho}\frac{|\nabla\eta|^2}{2} - \frac{\rho}{2\alpha}\left(\frac{\eta}{\rho} - 1\right)^2 \right) d\Omega$$

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$\Longrightarrow$ Time to derive the Euler-Lagrange equations !

Hints on calculus of variations

$$\tilde{\mathcal{L}} = \int_{\Omega_t} \left(\rho \frac{|\mathbf{u}|}{2}^2 - \frac{\rho}{2}^2 - \frac{1}{4\rho}\frac{|\nabla \eta|}{2}^2 - \frac{\rho}{2\alpha}\left(\frac{\eta}{\rho} - 1\right)^2 \right) d\Omega$$

$$\tilde{\mathcal{L}}(\overbrace{\mathbf{u}, \rho}^{\delta \mathbf{x}}, \underbrace{\eta, \nabla \eta}_{\delta \eta}) \Rightarrow \text{Two Euler-Lagrange equations}$$

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- Virtual displacement of the continuum ($\delta \mathbf{x}$):

$$(\rho \mathbf{u})_t + \operatorname{div} \left(\rho \mathbf{u} \otimes \mathbf{u} + \frac{1}{4\rho} \nabla \eta \otimes \nabla \eta \right) + \nabla \left(\frac{\rho^2}{2} - \frac{|\nabla \eta|^2}{4\rho} + \frac{\eta}{\alpha} \left(1 - \frac{\eta}{\rho} \right) \right) = 0$$

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- η variation ($\delta \eta$) :

$$\frac{1}{4\rho^2} \nabla \rho \cdot \nabla \eta - \frac{1}{4\rho} \Delta \eta = \frac{1}{\alpha} \left(1 - \frac{\eta}{\rho} \right)$$

Preliminary system

Thus the system of governing equations now writes :

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Idea : Include $\dot{\eta}$ into the Lagrangian !

Augmented Lagrangian - Attempt 2

Augmented Lagrangian approach

$$\tilde{\mathcal{L}}(\mathbf{u}, \rho, \eta, \nabla\eta, \dot{\eta}) \qquad \alpha, \beta \ll 1$$

$$\tilde{\mathcal{L}} = \int_{\Omega_t} \left(\rho \frac{|\mathbf{u}|^2}{2} - \frac{\rho^2}{2} - \frac{1}{4\rho} \frac{|\nabla\eta|^2}{2} - \frac{\rho}{2\alpha} \left(\frac{\eta}{\rho} - 1 \right)^2 + \frac{\beta\rho}{2} \dot{\eta}^2 \right) d\Omega$$

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Order reduction

1. We denote $w = \dot{\eta}$. Thus :

$$w = \eta_t + \mathbf{u} \cdot \nabla \eta \quad \Longrightarrow \quad \boxed{(\rho\eta)_t + \operatorname{div}(\rho\eta\mathbf{u}) = \rho w}$$

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Important : $\mathbf{p}(\mathbf{x}, t = 0) = \nabla \eta(\mathbf{x}, t = 0)$

Augmented NLS system

$$
\begin{cases}
\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \\
(\rho \mathbf{u})_t + \operatorname{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + \left(\frac{\rho^2}{2} - \frac{|\mathbf{p}|^2}{4\rho} + \frac{\eta}{\alpha}(1 - \frac{\eta}{\rho})\right) \mathbf{Id} + \frac{1}{4\rho} \mathbf{p} \otimes \mathbf{p}\right) = 0 \\
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(\rho\eta)_t + \operatorname{div}(\rho\eta\mathbf{u}) = \rho w \\
\mathbf{p}_t + \operatorname{div}\left((\mathbf{p} \cdot \mathbf{u} - w)\, \mathbf{I_d}\right) = 0, \qquad (\mathbf{curl}(\mathbf{p}) = 0)
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- Main question : Is this system hyperbolic ?

Augmented NLS system

$$
\begin{cases}
\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \\
(\rho \mathbf{u})_t + \operatorname{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + \left(\frac{\rho^2}{2} - \frac{|\mathbf{p}|^2}{4\rho} + \frac{\eta}{\alpha}(1 - \frac{\eta}{\rho})\right) \mathbf{Id} + \frac{1}{4\rho} \mathbf{p} \otimes \mathbf{p}\right) = 0 \\
(\rho w)_t + \operatorname{div}\left(\rho w \mathbf{u} - \frac{1}{4\beta\rho}\mathbf{p}\right) = \frac{1}{\alpha\beta}\left(1 - \frac{\eta}{\rho}\right) \\
(\rho\eta)_t + \operatorname{div}(\rho\eta\mathbf{u}) = \rho w \\
\mathbf{p}_t + \operatorname{div}\left((\mathbf{p}\cdot\mathbf{u} - w)\, \mathbf{I_d}\right) = 0, \qquad (\mathbf{curl}(\mathbf{p}) = 0)
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$\Rightarrow$ Strongly hyperbolic in one dimension of space.

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- Main question : Is this system hyperbolic ?

$\Rightarrow$ Strongly hyperbolic in one dimension of space.

$\Rightarrow$ Weakly hyperbolic in multi-dimensions. **+ curl constraint**

$\Rightarrow$ Strongly hyperbolic upgrade proposed in *Busto & al. 2021*.

Hyperbolicity of augmented NLS

1-d case: $\mathbf{u} = (u, 0, 0)^T$ and $\mathbf{p} = (p, 0, 0)^T$:

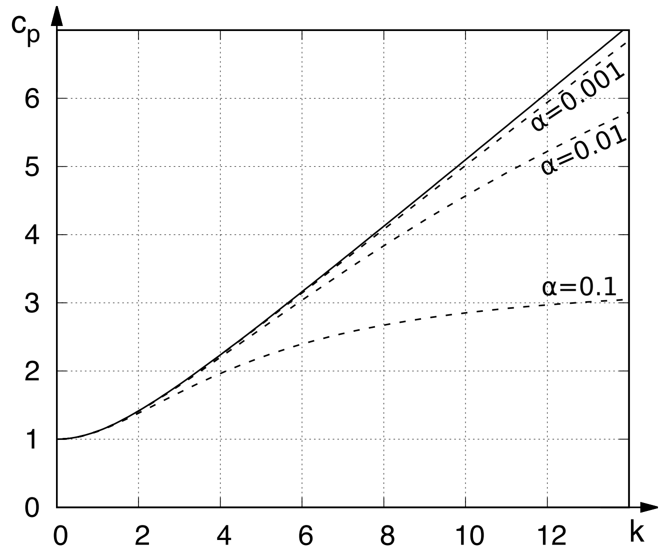
$$\mathbf{U}_t + A(\mathbf{U})\mathbf{U}_x = \mathbf{S}(\mathbf{U}) :$$

Eigensystem of A :

$$\begin{aligned}
\xi_1 &= u &&, \quad \mathbf{v_1} = \left(\tfrac{\rho}{\alpha\rho^3+\eta^2}, 0, 0, \tfrac{p}{\alpha\rho^3+\eta^2}, \tfrac{1}{2\eta-\rho}\right)^T \\
\xi_2 &= u + \tfrac{1}{2\rho\sqrt{\beta}} &&, \quad \mathbf{v_2} = (0, 0, \sqrt{\beta}, 2, 0)^T \\
\xi_3 &= u - \tfrac{1}{2\rho\sqrt{\beta}} &&, \quad \mathbf{v_3} = (0, 0, -\sqrt{\beta}, 2, 0)^T \\
\xi_4 &= u + \sqrt{\rho + \tfrac{\eta^2}{\alpha\rho^2}} &&, \quad \mathbf{v_4} = \left(\rho, \sqrt{\rho + \tfrac{\eta^2}{\alpha\rho^2}}, 0, p, 0\right)^T \\
\xi_5 &= u - \sqrt{\rho + \tfrac{\eta^2}{\alpha\rho^2}} &&, \quad \mathbf{v_5} = \left(\rho, -\sqrt{\rho + \tfrac{\eta^2}{\alpha\rho^2}}, 0, p, 0\right)^T
\end{aligned}$$

$\Rightarrow$ **The system is always hyperbolic.**

Dispersion relation comparison



The dispersion relation $c_p = f(k)$ for the original model (continuous line) and the dispersion relation for the Augmented model (dashed lines) for different values of λ and for $\beta = 10^{-4}$

Numerical scheme: IMEX-Type

1-d system of equations to solve :

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S}(\mathbf{U})$$

The idea is to solve the hyperbolic part explicitly and the source term implicitly in time according to the scheme ($\gamma = 1 - \frac{\sqrt{2}}{2}$):

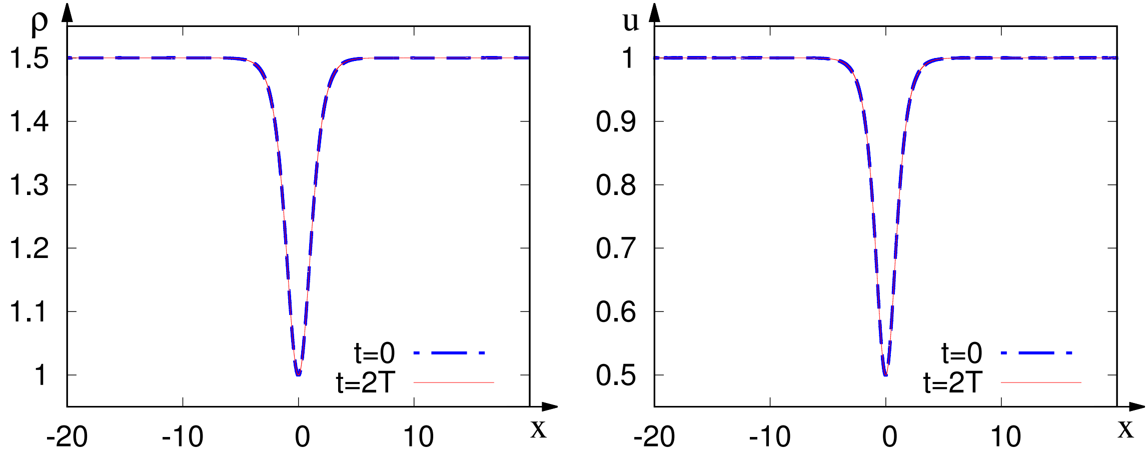
$$\mathbf{U}^\star = \mathbf{U}^n - \gamma \frac{\Delta t}{\Delta x}\left(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}\right) + \gamma \Delta t \mathbf{S}(\mathbf{U}^\star)$$

$$\mathbf{U}^{n+1} = \mathbf{U}^n - (\gamma - 1)\frac{\Delta t}{\Delta x}\left(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}\right) - (2-\gamma)\frac{\Delta t}{\Delta x}\left(F^\star_{i+\frac{1}{2}} - F^\star_{i-\frac{1}{2}}\right)$$
$$+ (1-\gamma)\Delta t S(\mathbf{U}^\star) + \gamma \Delta t S(\mathbf{U}^{n+1})$$

- MUSCL reconstruction in space.
- Rusanov solver for the fluxes.

Grey Soliton solution

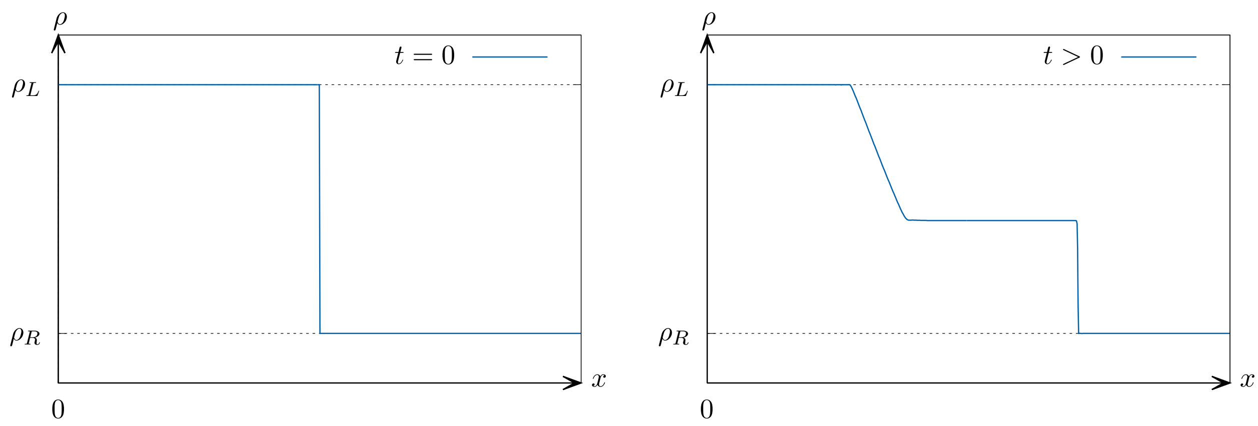
$$\rho(x,t) = b_1 - \frac{b_1 - b_3}{\cosh^2\left(\sqrt{b_1 - b_3}\,(x - Ut)\right)} \qquad u(x,t) = U - \frac{b_1\sqrt{b_3}}{\rho(x,t)}$$



Numerical profiles of ρ (left) and u (right) at $t = 0$ and $t = 2T$. The used domain is $L = [-20, 20]$ with $N = 100000$. Parameters used for the simulation are $U = 2, \beta = 10^{-4}, \alpha = 0.002$.

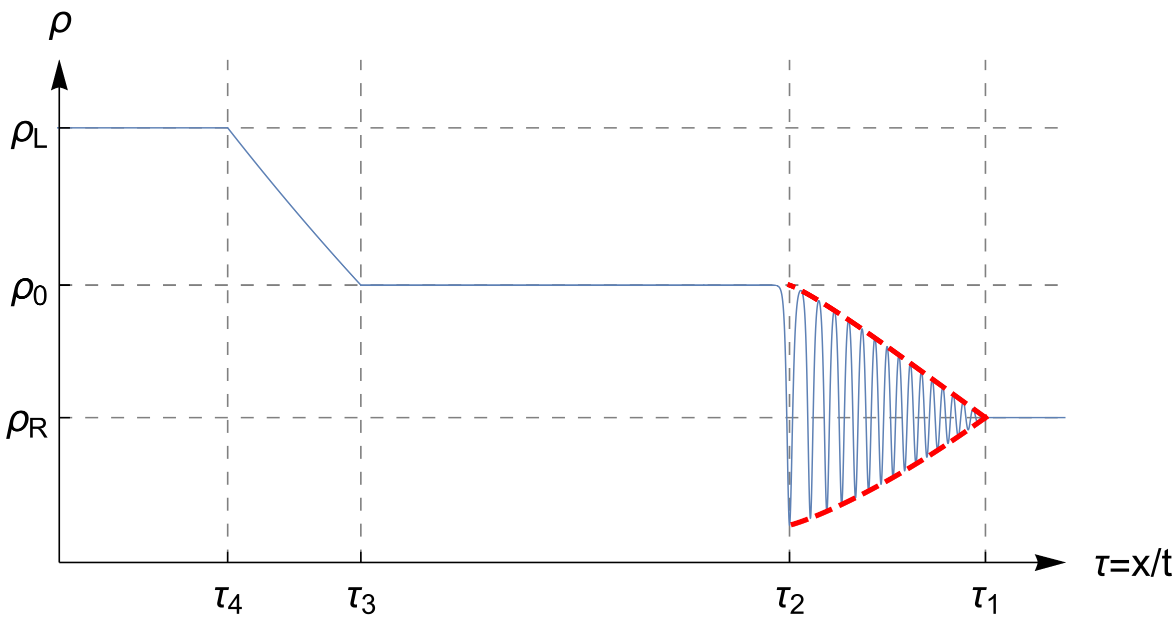
Shock waves for Euler equations

Riemann problem in dispersionless hydrodynamics governed by Euler Equations :



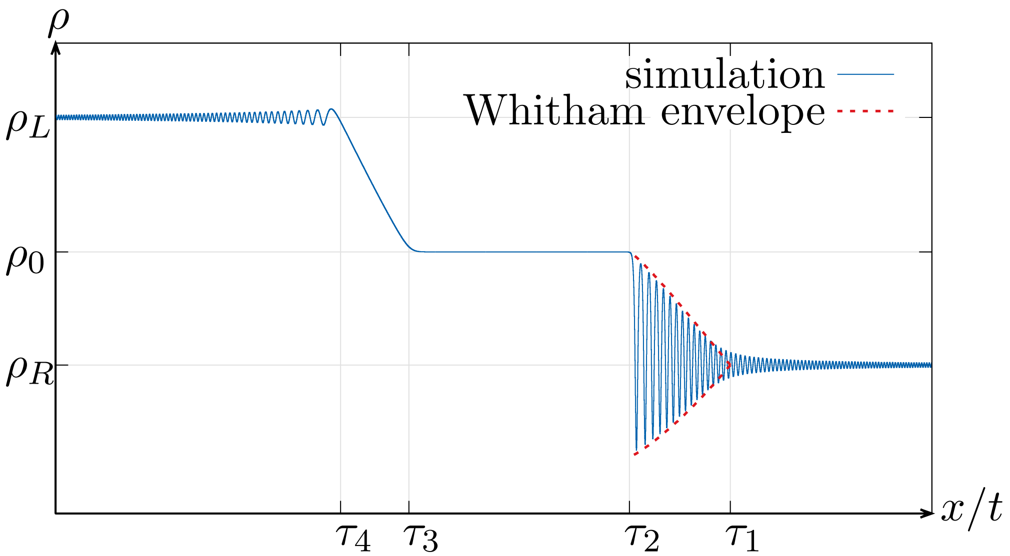
Shockwave solution to a Riemann problem for Euler Equations.

Dispersive Shock waves



Asymptotic profile of the solution to NLS equation (continuous line) for the Riemann problem $\rho_L = 2$, $\rho_R = 1$, $u_L = u_R = 0$. Oscillations shown at t=70

DSW Numerical results : ρ

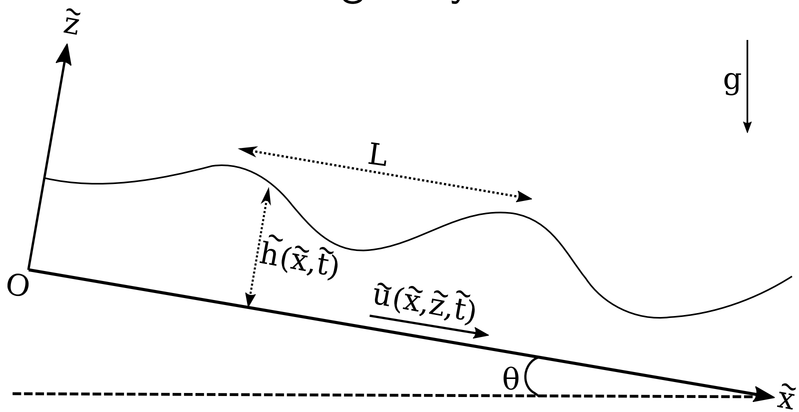


Comparison of the numerical result (ρ) with the Whitham modulational profile of the DSW at $t = 70$. $\beta = 2.10^{-5}$, $\alpha = 10^{-3}$, $N = 100000$. The computational domain is $[-500, 500]$

Thin film equations

Equations for thin films flow

Consider a thin film of liquid on an inclined horizontal plane which is moving under the effects of gravity :



Sketch of the setting.

Thin film flows $\rightarrow$ Long wave approximation : $\varepsilon = \frac{h}{L} \ll 1$

Governing equations

The dimensionless governing equations are obtained from Navier-Stokes equations as a leading order expansion in the long wave parameter ε [Lavalle,2015]:

$$h_t + (hU)_x = 0$$

$$(hU)_t + \left(hU^2 + \frac{\cos\theta}{2F^2}h^2 + \frac{2\lambda^2}{225}h^5\right)_x = \frac{\varepsilon\kappa_1}{F^2}hh_{xxx} + \frac{1}{\varepsilon Re}\left(\lambda h - \frac{3U}{h}\right) + o(\varepsilon)$$

$$Re = \frac{\tilde{h}_N\tilde{U}_N}{\nu},\ F = \frac{\tilde{U}_N}{\sqrt{g\tilde{h}_N}},\ We = \frac{\rho}{\gamma}\tilde{h}_N\tilde{U}_N^2,\ \lambda = \frac{Re\sin\theta}{F^2},\ \kappa_1 = \frac{\varepsilon F^2}{We}$$

Equivalent system in dimensional variables :

$$h'_{t'} + (h'U')_{x'} = 0$$

$$(h'U')_{t'} + \left(h'U'^2 + \frac{gh'^2}{2}\cos\theta + \frac{g^2\sin(\theta)^2}{\nu^2}h'^5\right)_{x'} - \frac{\sigma}{\rho}h'h'_{x'x'x'} = gh'\sin(\theta) - \nu\frac{U'}{h'}$$

Considered system

$$\begin{cases} h_t + (hU)_x = 0 \\ (hU)_t + \left(hU^2 + \dfrac{2\lambda^2}{225} h^5 + \dfrac{\cos\theta}{2F^2} h^2 \right)_x - \dfrac{\varepsilon \kappa_1}{F^2} h h_{xxx} = \dfrac{1}{\varepsilon Re} \left(\lambda h - \dfrac{3U}{h} \right) \end{cases}$$

Additional difficulties :

1. Model admits inherent dissipation.
2. New system should preserve asymptotics in ε.

Deriving the augmented system

$$\boxed{\text{Dissipative equations}}$$

$$\boxed{\text{dissipationless equations}} \quad + \quad \boxed{\text{Dissipative source terms}}$$

Deriving the augmented system

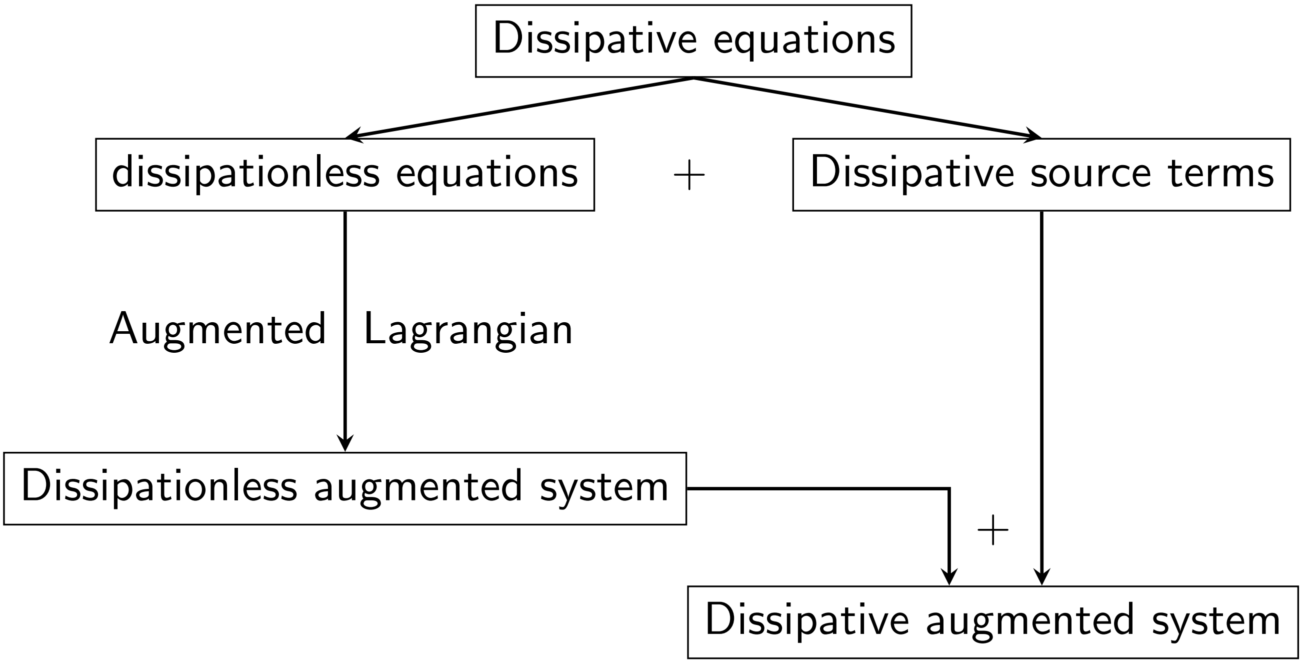
Dissipative equations

dissipationless equations + Dissipative source terms

Augmented Lagrangian

Dissipationless augmented system

Deriving the augmented system



Augmented system

After applying Hamilton's principle, the augmented system writes :

$$
\begin{aligned}
&h_t + (\rho u)_x = 0,\\
&(hu)_t + \left(hu^2 + \frac{h^2 cos\theta}{2F^2} + \frac{2\lambda^2 h^5}{225} + \frac{\varepsilon\kappa_1 p^2}{2F^2} + \frac{\eta}{\alpha}\left(1 - \frac{\eta}{h}\right)\right)_x = \frac{1}{\varepsilon Re}\left(\lambda h - \frac{3U}{h}\right)\\
&(h\eta)_t + (h\eta u)_x = \rho w,\\
&(hw)_t + \left(hwu - \frac{\varepsilon\kappa_1 p}{\beta F^2}\right)_x = \frac{1}{\alpha\beta}\left(1 - \frac{\eta}{\rho}\right),\\
&p_t + (pu - w)_x = 0
\end{aligned}
$$

$\implies$ How to choose α and β in this setting ?

Asymptotic study

- Too many small parameters : $\alpha, \beta, \varepsilon$.

$\Longrightarrow$ We pose $\alpha = \varepsilon^m$ and $\beta = \varepsilon^p$.

$\Longrightarrow$ Expand phase velocities of both systems in ε :

$$c_p = c_{p0} + \varepsilon c_{p_1} + \varepsilon^2 c_{p_2} + \ldots$$

then choose m and p so that :

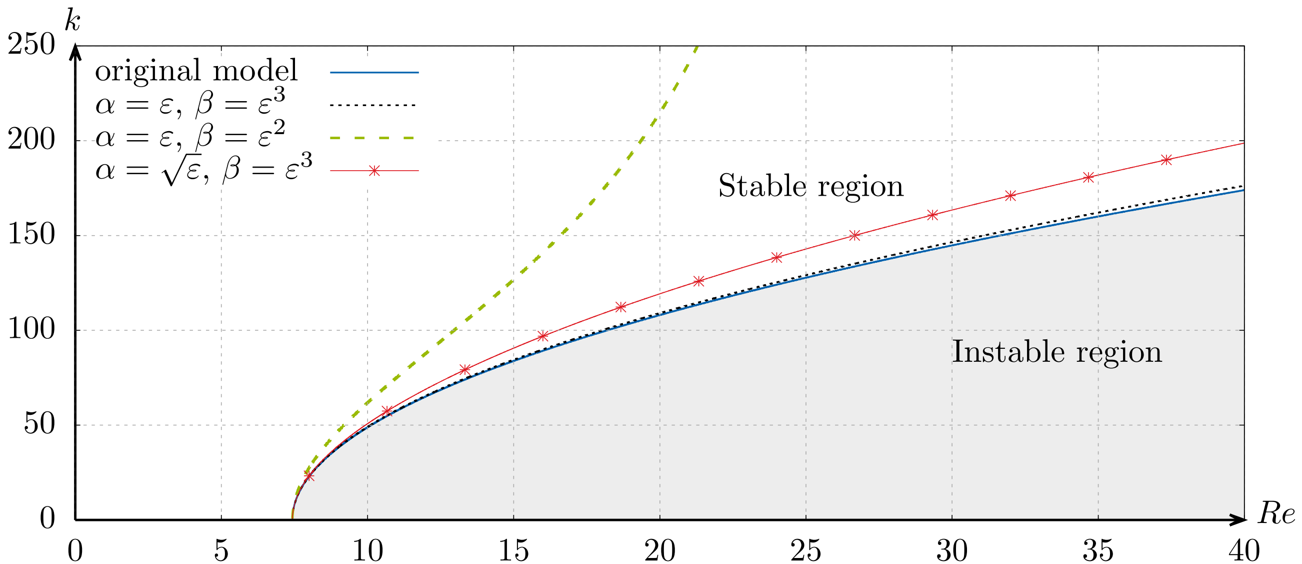
1. Phase speeds of both systems are consistent.
2. Neutral stability curves of both systems are consistent.

After tedious calculations

Phase speeds are consistent to 2nd order if $\alpha = o(\varepsilon)$ and $\beta = o(\varepsilon^3)$

What about stability analysis ?

Stability analysis



Neutral stability curves in the (k, Re) plane for the original model (blue continuous line) and the augmented model for various scalings of α and β with respect to ε. Parameters are $\theta = 6.4°$, $We = 0.184$, $F = 0.847$ and $\varepsilon = 0.006$.

One last step before results

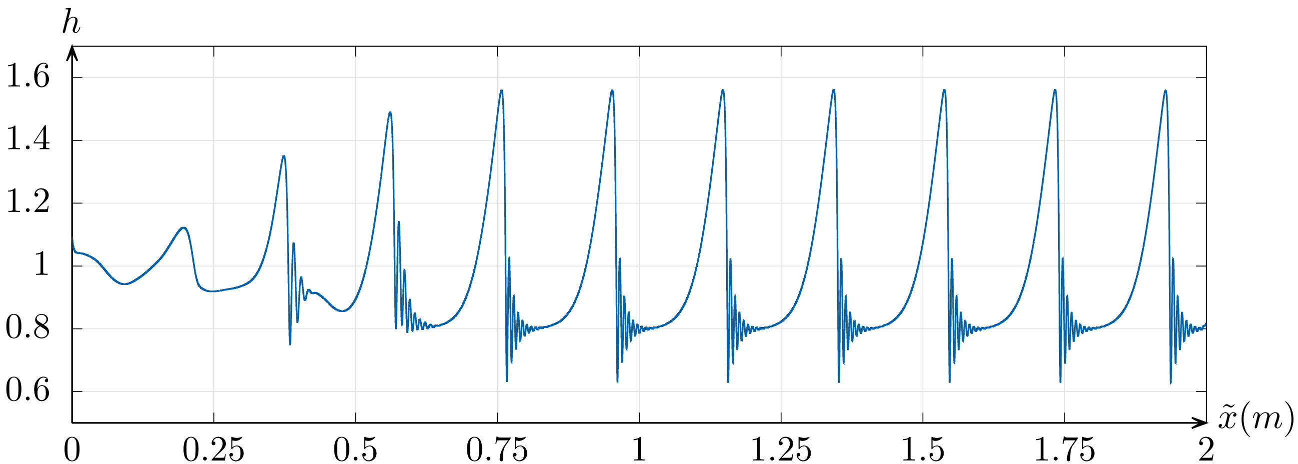
In order to compare with experimental results, 2nd order viscosity must be added to the model :

$$(hu)_t + \left(hu^2 + \frac{h^2}{2F^2}cos\theta + \frac{2\lambda^2 h^5}{225}\right)_x = \frac{1}{\varepsilon Re}\left(\lambda h - \frac{3u}{h}\right) + \frac{\varepsilon\kappa_1}{F^2}hh_{xxx} + \frac{9\varepsilon}{2Re}(hu_x)_x$$

On the discrete level, we use centered finite differences :

$$(hu_x)_x = \left((hu_x)_{i+\frac{1}{2}} - (hu_x)_{i-\frac{1}{2}}\right)/\Delta x$$
$$(hu_x)_{i+\frac{1}{2}} = h_{i+\frac{1}{2}}\left(u_{i+1} - u_{i-1}\right)/\Delta x$$

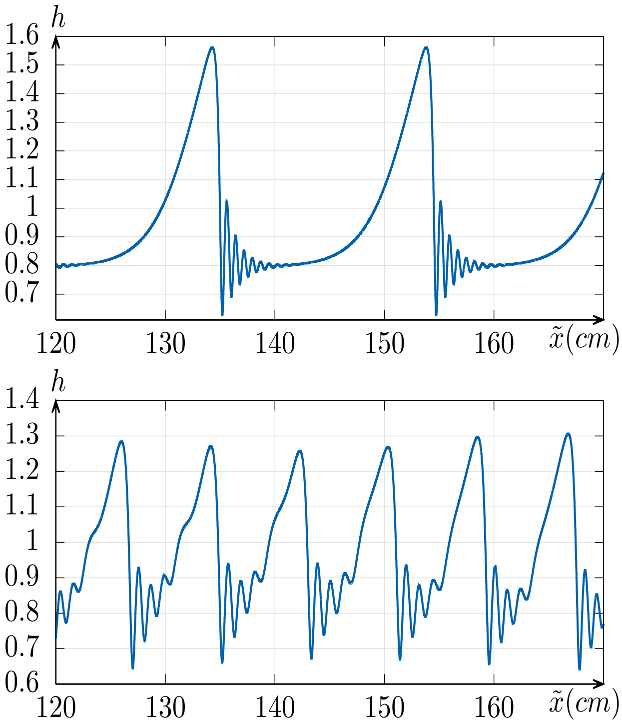
Liu & Gollub's experiment (1994)



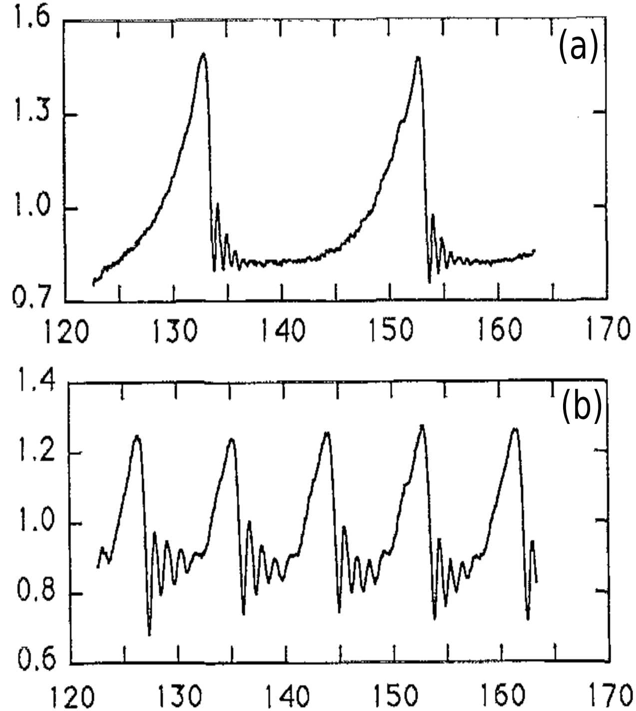
Dimensionless water height as a function of space (dimensioned), in the setting of the Liu & Gollub experiment, for an imposed frequency of $1.5Hz$. (Obtained through numerical simulation). Parameters used here are : $Re = 19.33$, $\kappa = 1.440.10^{-4}$, $Fr = 0.8476$,$\theta = 6.4°$

Results for different frequencies

Numerical simulation result

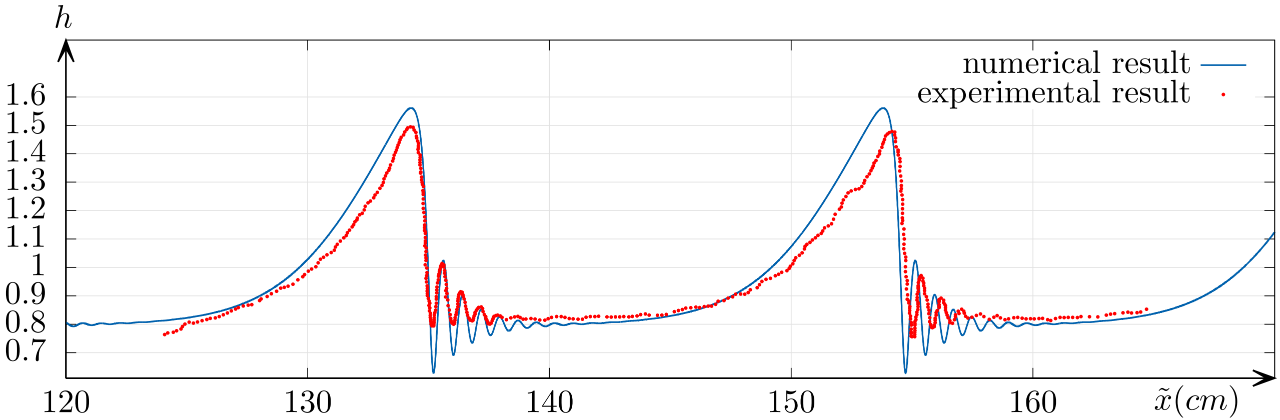


Experimental result



Inlet oscillation frequency : (a) 1.5Hz, (b) 3.0 Hz

Thorough comparison for $1.5Hz$



Superimposed numerical simulation with the experimental result for $\tilde{f} = 1.5Hz$.

$\Longrightarrow$ Very good agreement in both shape and values.

$\Longrightarrow$ Involved waveslengths are correctly captured by the model.

Test for a nonlinear surface tension

the same approach was used in the case of a nonlinear surface tension term :

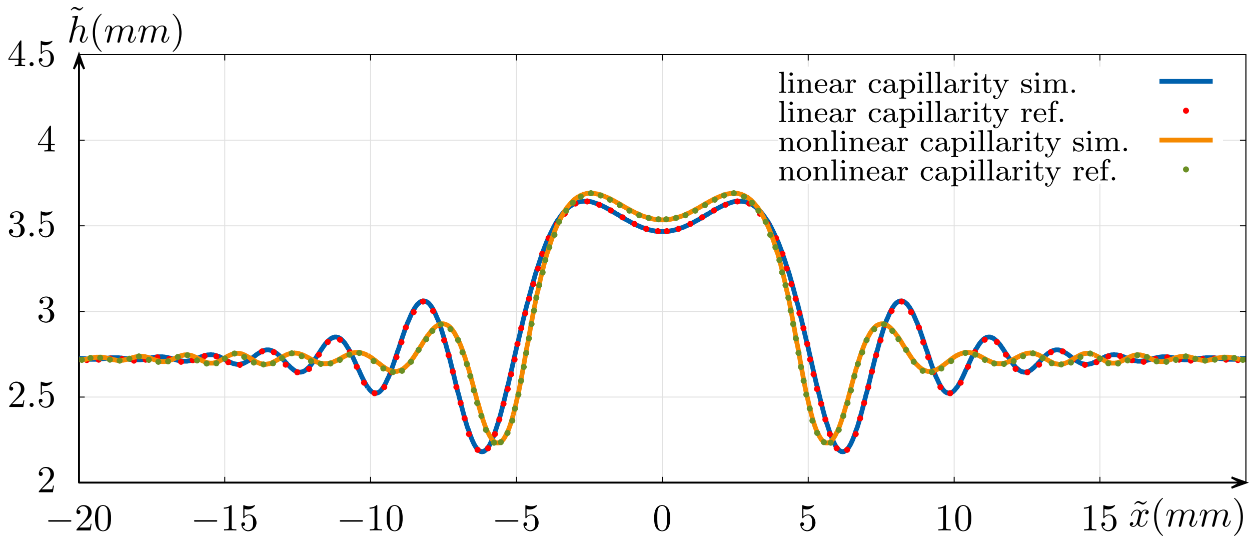
$$\mathcal{L} = \int_{\Omega_t} \left(\frac{1}{2}\tilde{h}\tilde{U}^2 - \frac{1}{2}g\tilde{h}^2 - \frac{\sigma}{\rho}\sqrt{1+\tilde{h}_{\tilde{x}}^2} \right) d\Omega$$

Corresponding augmented Lagrangian is given by :

$$\mathcal{L} = \int_{\Omega_t} \left(\frac{1}{2}\tilde{h}\tilde{U}^2 + \frac{1}{2}\tilde{\beta}\tilde{h}\tilde{w}^2 - \frac{1}{2}g\tilde{h}^2 - \frac{\sigma}{\rho}\sqrt{1+\tilde{p}^2} + \frac{\tilde{h}}{2\tilde{\alpha}}\left(1-\frac{\tilde{\eta}}{\tilde{h}}\right)^2 \right) d\Omega$$

$\Longrightarrow$ Corresponding system of equations is shown hyperbolic.

Results



Comparison of the obtained numerical results (solid lines) with the converged numerical solutions shown in *Bresch et.al [2020]* (dots), at $t = 5ms$. Parameters used here are $g = 9.81m.s^{-2}$, $\sigma = 0.0728Kg.s^{-2}$, $\rho = 1000Kg.m^{-3}$, $h_0 = 2.725mm$. $\tilde{\alpha} = 10^{-3}m^{-2}s^2$ and $\tilde{\beta} = 10^{-5}$. Results are shown with a mesh resolution of $n = 5000$.

Preliminary conclusion

- A generic approach to build a first order hyperbolic approximation of Euler-Korteweg equations was developed.
- Analytical, asymptotic and numerical comparisons between the original and augmented system were done.
- Numerical results have shown very good agreement for two specific systems : Defocusing NLS and thin films flows in stationary and non stationary cases.
- Extension to nonlinear forms of capillary term was shown successful.

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✗ Model is not perfectly hyperbolic for viscous flows.

Hyperbolic Navier-Stokes-Korteweg equations

Navier-Stokes-Korteweg systems

Viscous tensor is added to Euler-Korteweg equations:

$$\begin{cases} \rho_t + \mathrm{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \underline{\underline{K}} + \underline{\underline{S}} \end{cases}$$

where the (dispersive) Korteweg stress tensor is given by:

$$\underline{\underline{K}} = \rho \nabla \left(K(\rho)\Delta\rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2 \right)$$

and the (viscous) Navier-Stokes stresses are given by

$$\underline{\underline{S}} = \mu \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3} \mathrm{div}(\mathbf{u}) \mathbf{I} \right)$$

General difficulties with NSK system

Main difficulties are given by

1. Nonlinear High-order terms.
2. Very constricting CFL time-stepping.
3. Often coupled with non-convex equations of state (unphysical negative pressure), for example

$$p = \frac{\rho RT}{1 - b\rho} - a\rho^2, \qquad a > 0,\ b > 0$$

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- A hyperbolic reformulation of Navier-Stokes Equations (Godunov-Peshkov-Romenski) was presented in the previous lecture.
- **The idea : Combine both models.**

Hyperbolic NSK = Hyperbolic EK + Hyperbolic NS

Ma.C	$\rho_t + \mathrm{div}(\rho \mathbf{u}) = 0$
Mo.B	$(\rho \mathbf{u})_t + \mathrm{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + \mathbf{PId} + K(\rho)\mathbf{p} \otimes \mathbf{p} - \sigma\right) = 0$
η evolution	$(\rho\eta)_t + \mathrm{div}(\rho\eta\mathbf{u}) = \rho w$
η − E-L	$(\rho w)_t + \mathrm{div}\left(\rho w \mathbf{u} - \frac{K(\rho)}{\beta}\mathbf{p}\right) = \frac{\lambda}{\beta}\left(1 - \frac{\eta}{\rho}\right)$
p evolution	$\mathbf{p}_t + \nabla\left(\mathbf{p}\cdot\mathbf{u} - w\right) = 0$
Distortion	$\mathbf{A}_t + \nabla(\mathbf{A}\mathbf{u}) + \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}}\right)^T\right)\cdot \mathbf{u} = 0$
HM	$(\rho \mathbf{J})_t + \mathrm{div}(\rho\mathbf{J} \otimes \mathbf{u} + T\mathbf{I}) = -\rho \mathbf{H}$

with $\sigma = -\rho \mathbf{A}^T E_{\mathbf{A}}$ and $\mathbf{P}$ is the hyperbolic Korteweg stress.

GLM-curl cleaning approach

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In order to avoid spurious errors on $\mathbf{curl}(\mathbf{p})$ we use GLM-curl cleaning [3]:

$$\mathbf{p}_t - \nabla w + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T \mathbf{p} + \left(\frac{\partial \mathbf{p}}{\partial \mathbf{x}}\right) \mathbf{u} + 2a_c\rho\, \mathbf{curl}(\psi) = 0$$

$$\psi_t + \left(\frac{\partial \psi}{\partial \mathbf{x}}\right)^T \mathbf{u} - \frac{a_c}{2\rho}\mathbf{curl}(\mathbf{p}) = 0$$

with fast cleaning speed a_c that propagates $\mathbf{curl}(\mathbf{p})$ errors.

About non-convex energy

Take barotropic Euler equations for example. Energy is given by :

$$E = W(\rho) + \frac{1}{2}\rho|\mathbf{u}|^2$$

Eigenvalues of the PDE are given by :

$$\lambda_1 = u - c, \quad \lambda_2 = u, \quad \lambda_3 = u + c$$

where c is given by : $c = \sqrt{\rho W''(\rho)}$.
$\Longrightarrow$ Complex values for non-convex energy!.

Recall Hyperbolic NLS equation eigenvalues

$$\begin{aligned}
\xi_1 &= u &,\quad \mathbf{v_1} &= \left(\tfrac{\rho}{\alpha\rho^3+\eta^2}, 0, 0, \tfrac{p}{\alpha\rho^3+\eta^2}, \tfrac{1}{2\eta-\rho}\right)^T \\
\xi_2 &= u + \tfrac{1}{2\rho\sqrt{\beta}} &,\quad \mathbf{v_2} &= (0, 0, \sqrt{\beta}, 2, 0)^T \\
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\end{aligned}$$

Recall Hyperbolic NLS equation eigenvalues

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- Fast dispersive characteristics cover-up for non-convex regions!

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- Fast dispersive characteristics cover-up for non-convex regions!
- Restores hyperbolicity even for non-convex internal energy.

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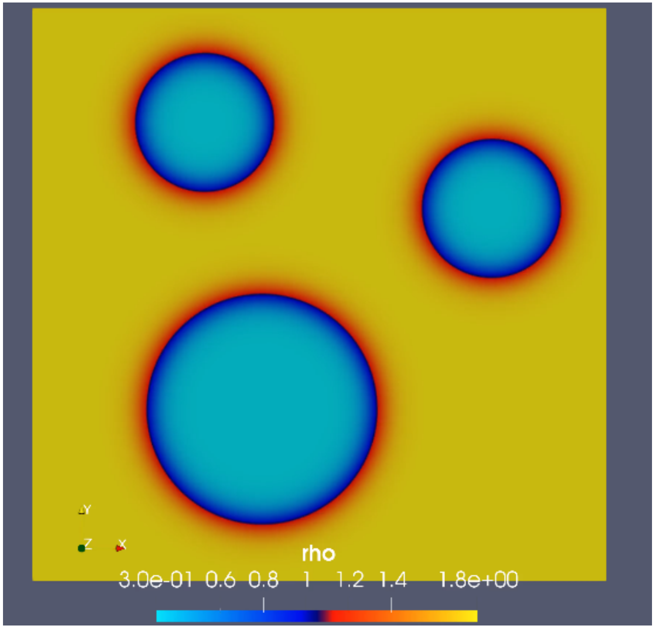
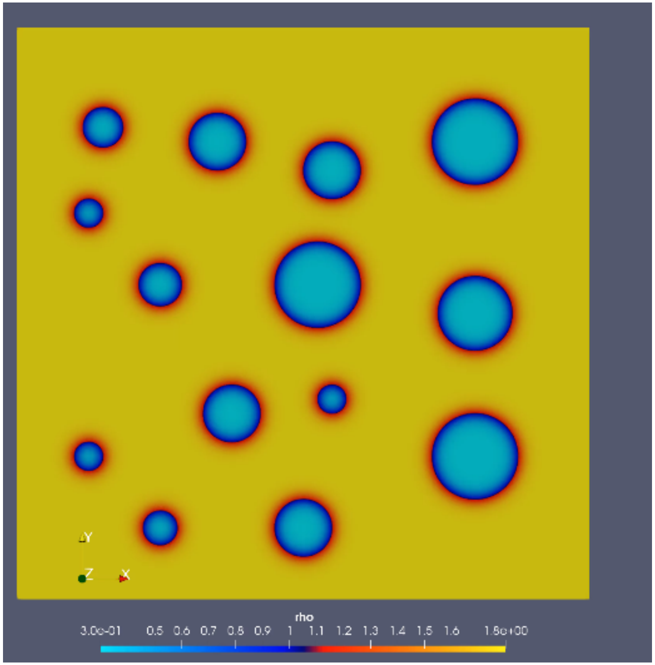
5. generally requires usual linear in Δx time-stepping.

Ostwald Ripening with 3 bubbles



Ostwald ripening with three bubbles (Obtained with a P_3P_3 ADER-DG scheme + Periodic boundary conditions + WENO3 subcell limiting on a 96×96 grid)

Ostwald Ripening with 14 bubbles



Ostwald ripening test case with 14 bubbles (Obtained with a P_3P_3 ADER-DG scheme + Periodic boundary conditions + WENO3 subcell limiting on a 192×192 grid)

Conclusion & Perspectives

Summary :

- A first order hyperbolic reformulation of general dispersive and diffusive continuum mechanics equations is presented
- Particular attention was given to Navier-Stokes-Korteweg equations.

Current concerns :

- Flux splitting approaches to separate fast characteristics from the rest.
- Extension to more general and nonlinear energies depending on other forms of the gradient of macroscopic variables.
- Better schemes for exactly conserving the curl-free and/or divergence-free constraints.

Thank you

Thank you for your attention !

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