A structure-preserving scheme for a hyperbolic approximation of the Navier-Stokes-Korteweg equations

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Joint work with
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January 16th, 2024

Main objective

Consider the Navier-Stokes-Korteweg system of equations :

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&\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \\
&(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = \mu \operatorname{div}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3}\operatorname{div}(\mathbf{u})\mathbf{I}\right) \\
&\qquad + \rho \nabla \left(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\right)
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Suggested solution

A first-order hyperbolic approximation to the NSK system!

A subset of connected works and topics

1. A family of Parabolic relaxation models of NSK equations.
   - ⇒ Diehl, Kremser, Kröner, Rohde 2016 (DG for NSK)
   - ⇒ Hitz,Keim,Munz,Rohde 2020 (Barotropic)
   - ⇒ Keim,Munz,Rohde 2023 [non-Isothermal NSK]

   and many other works...
2. Hyperbolic approximation of Euler-Korteweg equations.
   - ⇒ Dhaouadi, Favrie, Gavrilyuk 2019. (Schrödinger equation)
   - ⇒ Dhaouadi, Gavrilyuk, Vila 2022. (Thin films).
   - ⇒ Bourgeois, Lombard, Favrie 2020 (Solids with nonconvex EOS)
   - ⇒ Bresch *et al.*,2020 (2nd Order Hyperbolic)
3. Hyperbolic reformulation of Navier-Stokes equations.
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Idea

Combine our augmented Lagrangian model with the general *GPR* model of continuum mechanics.

Outline

Dissipationless Euler-Korteweg equations

The equations write :

$$\begin{cases} \rho_t + \mathrm{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = \rho \nabla \left(K(\rho) \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2 \right) \end{cases}$$

where $\rho = \rho(\mathbf{x}, t)$, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and $(\mathbf{x}, t) \in \mathbb{R}^d \times [0, T]$

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- $K(\rho) = \gamma$: **Compressible flow with surface tension**

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- $K(\rho) = \frac{1}{4\rho}$: **Quantum hydrodynamics**

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Lagrangian for the Euler-Korteweg system

(EK) system can be derived from the Lagrangian :

$$\mathcal{L} = \int_{\Omega_t} \left(\frac{\rho\,|\mathbf{u}|^2}{2} - W(\rho) - \gamma \frac{|\nabla \rho|^2}{2} \right) d\Omega$$

↓ Variational principle
+
Differential constraint : $\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0$

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla(P(\rho)) = \gamma \rho \nabla(\Delta \rho)$$

with $P(\rho) = \rho W'(\rho) - W(\rho)$

Augmented Lagrangian approach

$$\mathcal{L}(\mathbf{u},\rho,\nabla\rho)=\int_{\Omega_t}\left(\frac{1}{2}\rho\left|\mathbf{u}\right|^2-W(\rho)-\gamma\frac{\left|\nabla\rho\right|^2}{2}\right)d\Omega$$

$$\rho_t+\operatorname{div}(\rho\mathbf{u})=0$$

'Augmented' Lagrangian approach [Favrie-Gavrilyuk 2017]

$$\tilde{\mathcal{L}}(\mathbf{u},\rho,\eta,\nabla\eta)\qquad(\eta\longrightarrow\rho)$$

$$\tilde{\mathcal{L}}=\int_{\Omega_t}\left(\rho\frac{\left|\mathbf{u}\right|^2}{2}-W(\rho)-\gamma\frac{\left|\nabla\eta\right|^2}{2}-\frac{1}{2\alpha\rho}\left(\rho-\eta\right)^2\right)d\Omega$$

$\frac{1}{2\alpha\rho}\left(\rho-\eta\right)^2$: Classical Penalty term

Preliminary system

This is the system we have

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla (P(\rho)) = \operatorname{div}(\mathbf{K}_\alpha) \\ -\gamma \Delta \eta = \frac{1}{\alpha}\left(1 - \frac{\eta}{\rho}\right) \end{cases}$$

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Idea : Include $\dot{\eta}$ into the Lagrangian !

Augmented Lagrangian - Attempt 2

Augmented Lagrangian approach

$$\tilde{\mathcal{L}}(\mathbf{u}, \rho, \eta, \nabla\eta, \dot{\eta}) \qquad \alpha, \beta \ll 1$$

$$\tilde{\mathcal{L}} = \int_{\Omega_t} \left(\rho \frac{|\mathbf{u}|^2}{2} - W(\rho) - \frac{\gamma}{2}|\nabla\eta|^2 - \frac{1}{2\alpha\rho}(\rho - \eta)^2 + \frac{\beta\rho}{2}\dot{\eta}^2 \right) d\Omega$$

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⇒ There are still high-order derivatives!

⇒ No time evolution for η!

Order reductions

1. We take $w = \dot{\eta}$ as independent variable. Thus:

$$w = \eta_t + \mathbf{u} \cdot \nabla \eta \quad \Longrightarrow \quad \boxed{(\rho\eta)_t + \operatorname{div}(\rho\eta\mathbf{u}) = \rho w}$$

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Important note

Initial data must be such that:

$$\mathbf{p}(\mathbf{x}, 0) = \nabla \eta(\mathbf{x}, 0), \quad w(\mathbf{x}, 0) = \dot{\eta}(\mathbf{x}, 0)$$

Final form of the approximate Euler-Korteweg system

$$\begin{cases} \rho_t & + \operatorname{div}(\rho\mathbf{u}) = 0 \\ (\rho\mathbf{u})_t & + \operatorname{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + P(\rho)\mathbf{Id} - \mathbf{K}_\alpha\right) = 0 \\ (\rho w)_t & + \operatorname{div}\left(\rho w\mathbf{u} - \gamma\mathbf{p}\right) = \frac{1}{\alpha\beta}\left(1 - \frac{\eta}{\rho}\right) \\ \mathbf{p}_t & + \nabla(\mathbf{p}\cdot\mathbf{u} - w) \ = 0 \\ (\rho\eta)_t & + \operatorname{div}(\rho\eta\mathbf{u}) = \rho w \end{cases}$$

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$\Rightarrow$ Now the system is Gallilean invariant...

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$$\mathbf{K}_\alpha = \left(\frac{\gamma}{2}|\mathbf{p}|^2 - \frac{\eta}{\alpha}\left(1 - \frac{\eta}{\rho}\right)\right)\mathbf{Id} - \gamma \mathbf{p} \otimes \mathbf{p}$$

But recall that $\mathbf{p} = \nabla \eta \;\Rightarrow\; \nabla \times \mathbf{p} = 0$ ...

$\Rightarrow$ Now the system is Gallilean invariant... But is it hyperbolic ?

Hyperbolicity in 1-D

$\mathbf{A}$ admits 5 eigenvalues that can be expressed as follows :
Reminder ($P(\rho)$: hydrostatic pressure, $p = \eta_x$)

$$\xi = \begin{pmatrix} u \\ u + \sqrt{\psi_1 + \psi_2} \\ u + \sqrt{\psi_1 - \psi_2} \\ u - \sqrt{\psi_1 + \psi_2} \\ u - \sqrt{\psi_1 - \psi_2} \end{pmatrix} \text{ with } \begin{cases} \psi_1 = \frac{1}{2}(a^2 + a_\gamma^2 + a_\alpha^2 + a_\beta^2) \\ \psi_2 = \frac{1}{2}\sqrt{(a^2 + a_\gamma^2 + a_\alpha^2 - a_\beta^2)^2 + 4a_\beta^2 a_\gamma^2} \\ a = \sqrt{P'(\rho)}, \quad a_\gamma = \sqrt{\frac{\gamma}{\rho} p^2} \\ a_\alpha = \frac{\eta}{\rho\sqrt{\alpha}}, \quad a_\beta = \sqrt{\frac{\gamma}{\beta\rho}} \end{cases}$$

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a^2: adiabatic sound speed.

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a^2: adiabatic sound speed.
a_γ: wave speed due to capillarity .

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a^2: adiabatic sound speed.
a_γ: wave speed due to capillarity .
a_α and a_β: First and second relaxation speeds.

Hyperbolicity in 1-D

$\mathbf{A}$ admits 5 eigenvalues that can be expressed as follows :
Reminder ($P(\rho)$: hydrostatic pressure, $p = \eta_x$)

$$\xi = \begin{pmatrix} u \\ u + \sqrt{\psi_1 + \psi_2} \\ u + \sqrt{\psi_1 - \psi_2} \\ u - \sqrt{\psi_1 + \psi_2} \\ u - \sqrt{\psi_1 - \psi_2} \end{pmatrix} \text{ with } \begin{cases} \psi_1 = \frac{1}{2}(a^2 + a_\gamma^2 + a_\alpha^2 + a_\beta^2) \\ \psi_2 = \frac{1}{2}\sqrt{(a^2 + a_\gamma^2 + a_\alpha^2 - a_\beta^2)^2 + 4a_\beta^2 a_\gamma^2} \\ a = \sqrt{P'(\rho)}, \quad a_\gamma = \sqrt{\frac{\gamma}{\rho} p^2} \\ a_\alpha = \frac{\eta}{\rho\sqrt{\alpha}}, \quad a_\beta = \sqrt{\frac{\gamma}{\beta\rho}} \end{cases}$$

a^2: adiabatic sound speed. (negative in non-convex regions!!)
a_γ: wave speed due to capillarity .
a_α and a_β: First and second relaxation speeds.

Van der Waals equation of state

In the context of two-phase flows, the equation of state is non-convex

$$p = \frac{\rho R T}{1 - b\rho} - a\rho^2, \qquad a > 0,\ b > 0$$

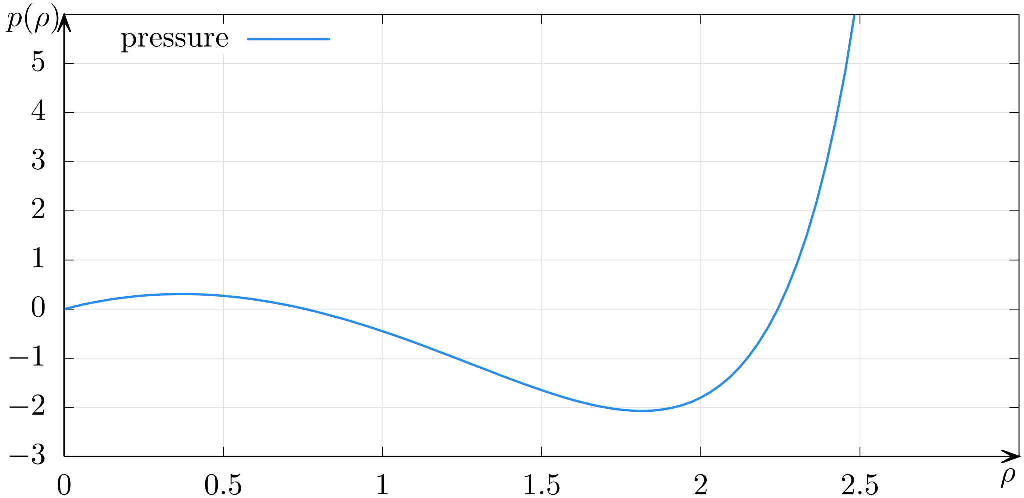


Figure 1: Van der Waals pressure for $T = 0.85, a = 3, b = 1/3, R = 8/3$

Navier-Stokes-Korteweg equations

In general, the equations write

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = \underline{\underline{S}} + \underline{\underline{K}} \end{cases}$$

where $\rho = \rho(\mathbf{x}, t)$, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and $(\mathbf{x}, t) \in \mathbb{R}^d \times [0, T]$
The (dispersive) Korteweg stress tensor is given by:

$$\underline{\underline{K}} = \rho \nabla \left(\gamma \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2 \right)$$

and the (viscous) Navier-Stokes stresses are given by

$$\underline{\underline{S}} = \mu \operatorname{div} \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3} \operatorname{div}(\mathbf{u}) \mathbf{I} \right)$$

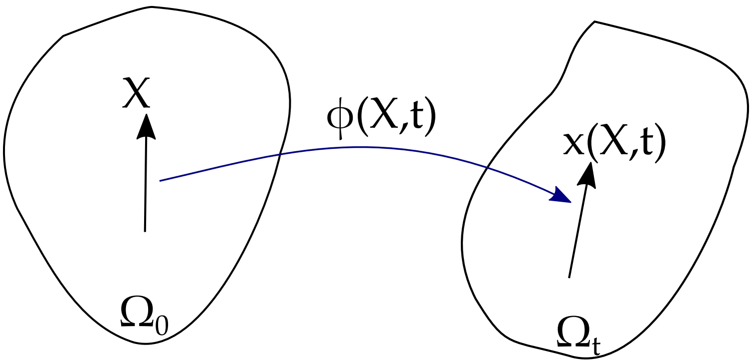
Godunov-Peshkov-Romenski Model of continuum mechanics

Deformation gradient:

$$\mathbf{F} = \left[\frac{\partial x_i}{\partial X_j}\right]$$

Inverse Deformation gradient:

$$\mathbf{A} = \mathbf{F}^{-1} = \left[\frac{\partial X_i}{\partial x_j}\right]$$



$$\partial_t(\mathbf{A}) + \nabla(\mathbf{A}\mathbf{u}) + \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}}\right)^T\right) \cdot \mathbf{u} = 0 \quad \text{(Solids)}$$

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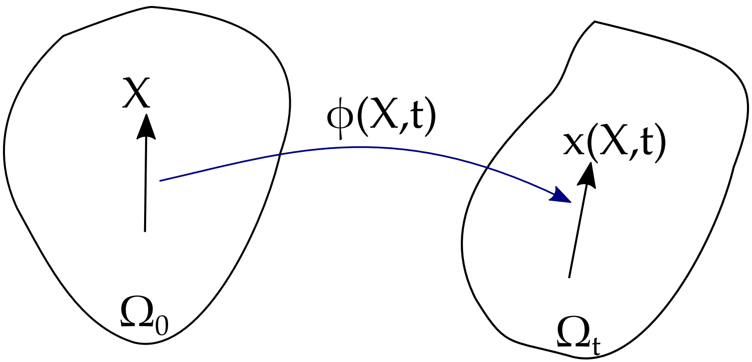

Inverse Deformation gradient:

$$\mathbf{A} = \mathbf{F}^{-1} = \left[\frac{\partial X_i}{\partial x_j}\right]$$

$$\partial_t(\mathbf{A}) + \nabla(\mathbf{A}\mathbf{u}) + \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}}\right)^T\right) \cdot \mathbf{u} = 0 \quad \text{(Solids)}$$

$$\partial_t(\mathbf{A}) + \nabla(\mathbf{A}\mathbf{u}) + \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}}\right)^T\right) \cdot \mathbf{u} = \frac{1}{\tau}\,\mathbf{S}(\mathbf{A}) \quad \text{(Fluids)}$$

Hyperbolic NSK = Hyperbolic EK + Hyperbolic NS

(Black: Euler part, Red: Dispersive part, Blue: Viscous part.)

$$\partial_t(\rho) + \mathrm{div}(\rho\mathbf{u}) = 0$$

$$\partial_t(\rho\mathbf{u}) + \mathrm{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + P(\rho)\mathbf{Id} - K_\alpha - \sigma\right) = 0$$

$$\partial_t(\rho\eta) + \mathrm{div}(\rho\eta\mathbf{u}) = \rho w$$

$$\partial_t(\rho w) + \mathrm{div}\left(\rho w\mathbf{u} - \gamma\mathbf{p}/\beta\right) = (\alpha\beta)^{-1}\left(1 - \eta/\rho\right)$$

$$\partial_t(\mathbf{p}) + \nabla\left(\mathbf{p}\cdot\mathbf{u} - w\right) + \left(\nabla\times\mathbf{p}\right)\times\mathbf{u} = 0,$$

$$\partial_t(\mathbf{A}) + \nabla(\mathbf{A}\mathbf{u}) + \left(\frac{\partial\mathbf{A}}{\partial\mathbf{x}} - \left(\frac{\partial\mathbf{A}}{\partial\mathbf{x}}\right)^T\right)\cdot\mathbf{u} = -\frac{3}{\tau}\det(\mathbf{A})^{5/3}\mathbf{A}\,\mathrm{dev}(\mathbf{G})$$

$$\text{where}\begin{cases}\sigma = -\rho c_s^2\mathbf{G}\,\mathrm{dev}(\mathbf{G}) \text{ where } \mathbf{G} = \mathbf{A}^T\mathbf{A}\\ \mathbf{K}_\alpha = -\gamma\mathbf{p}\otimes\mathbf{p} + \left(\frac{\gamma}{2}|\mathbf{p}|^2 - \frac{\eta}{\alpha}\left(1 - \frac{\eta}{\rho}\right)\right)\mathbf{Id}\end{cases}$$

Eigenvalues - Hyperbolicity

$\Rightarrow$ 18 Real Eigenvalues (Linearized around $A = \mathbf{I}, \mathbf{p} = (p1, 0, 0)^T$)

Transport: $\lambda_{1-10} = u_1$

shear waves: $\begin{cases} \lambda_{11-12} = u_1 + c_s, \\ \lambda_{13-14} = u_1 - c_s, \end{cases}$

Mixed waves:

$$\begin{cases} \lambda_{15} = u_1 - \sqrt{Z_1 + Z_2} \\ \lambda_{16} = u_1 - \sqrt{Z_1 - Z_2} \\ \lambda_{17} = u_1 + \sqrt{Z_1 + Z_2} \\ \lambda_{18} = u_1 + \sqrt{Z_1 - Z_2} \end{cases}, \begin{cases} Z_1 = \frac{1}{2}(a_0^2 + a_s^2 + a_\gamma^2 + a_\alpha^2 + a_\beta^2), \\ Z_2 = \sqrt{Z_1^2 - a_\beta^2(a_0^2 + a_\alpha^2 + a_s^2)}, \\ a_0 = \sqrt{\rho W''(\rho)}, \quad a_s = \sqrt{\frac{4}{3}c_s^2} \\ a_\alpha = \frac{\eta}{\rho\sqrt{\alpha}}, \quad a_\beta = \sqrt{\frac{\gamma}{\beta\rho}}, \quad a_\gamma = \sqrt{\frac{\gamma}{\rho}p_1^2} \end{cases}$$

System to be solved numerically

A set of classical conservation laws:

$$
\begin{aligned}
&\partial_t(\rho) && + \operatorname{div}(\rho\mathbf{u}) = 0 \\
&\partial_t(\rho\mathbf{u}) && + \operatorname{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + P(\rho)\mathbf{Id} - K_\alpha - \sigma\right) = 0 \\
&\partial_t(\rho\eta) && + \operatorname{div}(\rho\eta\mathbf{u}) = \rho w \\
&\partial_t(\rho w) && + \operatorname{div}\left(\rho w\mathbf{u} - \gamma\mathbf{p}/\beta\right) = (\alpha\beta)^{-1}\left(1 - \eta/\rho\right)
\end{aligned}
$$

A set of potentially curl constrained vectors:

$$
\begin{aligned}
&\partial_t(\mathbf{p}) && + \nabla\left(\mathbf{p}\cdot\mathbf{u} - w\right) = 0, \\
&\partial_t(\mathbf{A_1}) && + \nabla(\mathbf{A_1}\cdot\mathbf{u}) + (\nabla\times\mathbf{A}_1)\times\mathbf{u} = -\frac{1}{\tau}\mathbf{S}_1 \\
&\partial_t(\mathbf{A_2}) && + \nabla(\mathbf{A_2}\cdot\mathbf{u}) + (\nabla\times\mathbf{A}_2)\times\mathbf{u} = -\frac{1}{\tau}\mathbf{S}_2 \\
&\partial_t(\mathbf{A_3}) && + \nabla(\mathbf{A_3}\cdot\mathbf{u}) + (\nabla\times\mathbf{A}_3)\times\mathbf{u} = -\frac{1}{\tau}\mathbf{S}_3
\end{aligned}
$$

System to be solved numerically

A set of classical conservation laws: MUSCL-Hancock FV scheme

$$\begin{aligned}
\partial_t(\rho) \quad & + \operatorname{div}(\rho \mathbf{u}) = 0 \\
\partial_t(\rho \mathbf{u}) \quad & + \operatorname{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + P(\rho)\mathbf{Id} - K_\alpha - \sigma\right) = 0 \\
\partial_t(\rho \eta) \quad & + \operatorname{div}(\rho \eta \mathbf{u}) = \rho w \\
\partial_t(\rho w) \quad & + \operatorname{div}\left(\rho w \mathbf{u} - \gamma \mathbf{p}/\beta\right) = (\alpha\beta)^{-1}\left(1 - \eta/\rho\right)
\end{aligned}$$

A set of potentially curl constrained vectors:

$$\begin{aligned}
\partial_t(\mathbf{p}) \quad & + \nabla\left(\mathbf{p} \cdot \mathbf{u} - w\right) = 0, \\
\partial_t(\mathbf{A_1}) \quad & + \nabla(\mathbf{A_1} \cdot \mathbf{u}) + (\nabla \times \mathbf{A}_1) \times \mathbf{u} = -\frac{1}{\tau}\mathbf{S}_1 \\
\partial_t(\mathbf{A_2}) \quad & + \nabla(\mathbf{A_2} \cdot \mathbf{u}) + (\nabla \times \mathbf{A}_2) \times \mathbf{u} = -\frac{1}{\tau}\mathbf{S}_2 \\
\partial_t(\mathbf{A_3}) \quad & + \nabla(\mathbf{A_3} \cdot \mathbf{u}) + (\nabla \times \mathbf{A}_3) \times \mathbf{u} = -\frac{1}{\tau}\mathbf{S}_3
\end{aligned}$$

System to be solved numerically

A set of classical conservation laws: MUSCL-Hancock FV scheme

$$\begin{aligned}
\partial_t(\rho) \quad &+ \operatorname{div}(\rho\mathbf{u}) = 0 \\
\partial_t(\rho\mathbf{u}) \quad &+ \operatorname{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + P(\rho)\mathbf{Id} - K_\alpha - \sigma\right) = 0 \\
\partial_t(\rho\eta) \quad &+ \operatorname{div}(\rho\eta\mathbf{u}) = \rho w \\
\partial_t(\rho w) \quad &+ \operatorname{div}\left(\rho w\mathbf{u} - \gamma\mathbf{p}/\beta\right) = (\alpha\beta)^{-1}\left(1 - \eta/\rho\right)
\end{aligned}$$

A set of potentially curl constrained vectors: VIP Treatment

$$\begin{aligned}
\partial_t(\mathbf{p}) \quad &+ \nabla\left(\mathbf{p}\cdot\mathbf{u} - w\right) = 0, \\
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\partial_t(\mathbf{A_2}) \quad &+ \nabla(\mathbf{A_2}\cdot\mathbf{u}) + (\nabla\times\mathbf{A}_2)\times\mathbf{u} = -\frac{1}{\tau}\mathbf{S}_2 \\
\partial_t(\mathbf{A_3}) \quad &+ \nabla(\mathbf{A_3}\cdot\mathbf{u}) + (\nabla\times\mathbf{A}_3)\times\mathbf{u} = -\frac{1}{\tau}\mathbf{S}_3
\end{aligned}$$

Exactly curl-free scheme: Staggered Grid

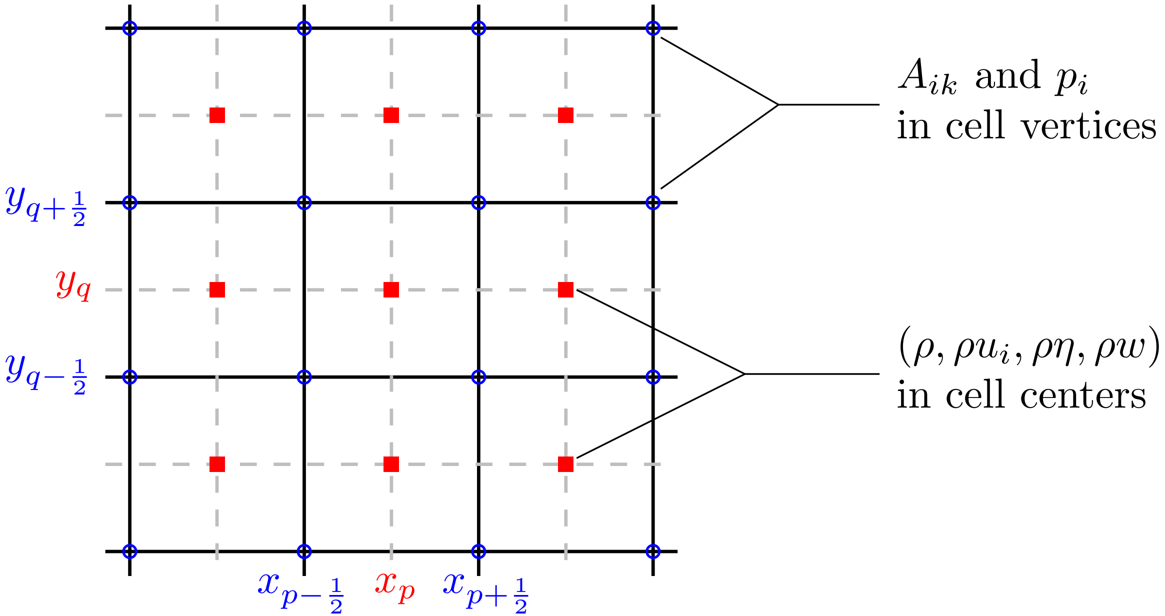
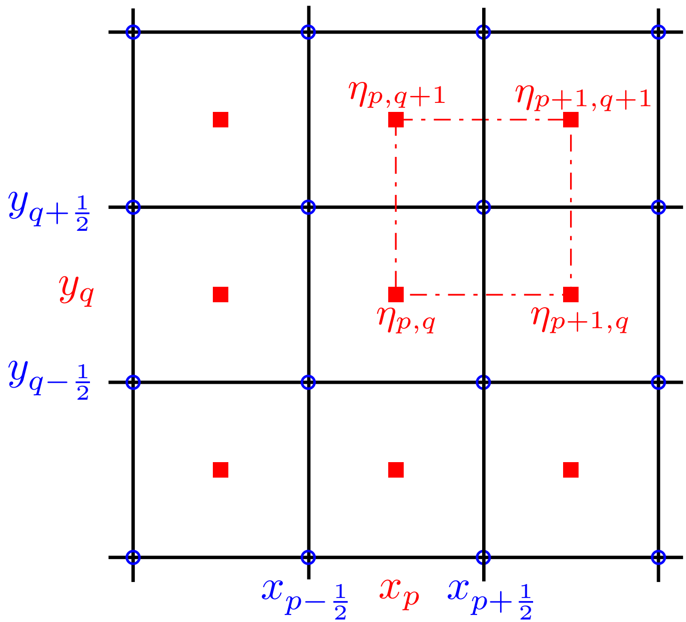


Figure 2: Schematic of the computational domain featuring the grid points and the staggered dual grid points. Red squares are barycenters and blue circles are the vertexes of the computational cells.

Exactly curl-free scheme: Gradient Stencil



$$\begin{cases} (\partial_x^h \phi)^{p+\frac{1}{2},q+\frac{1}{2}} = \dfrac{1}{2}\dfrac{\phi^{p+1,q} - \phi^{p,q}}{\Delta x} \\ \qquad\qquad + \dfrac{1}{2}\dfrac{\phi^{p+1,q+1} - \phi^{p,q+1}}{\Delta x}, \\ (\partial_y^h \phi)^{p+\frac{1}{2},q+\frac{1}{2}} = \dfrac{1}{2}\dfrac{\phi^{p,q+1} - \phi^{p,q}}{\Delta y} \\ \qquad\qquad + \dfrac{1}{2}\dfrac{\phi^{p+1,q+1} - \phi^{p+1,q}}{\Delta y}. \end{cases}$$

Figure 3: Stencil of the gradient field computed in every corner

Exactly curl-free scheme: Curl stencil

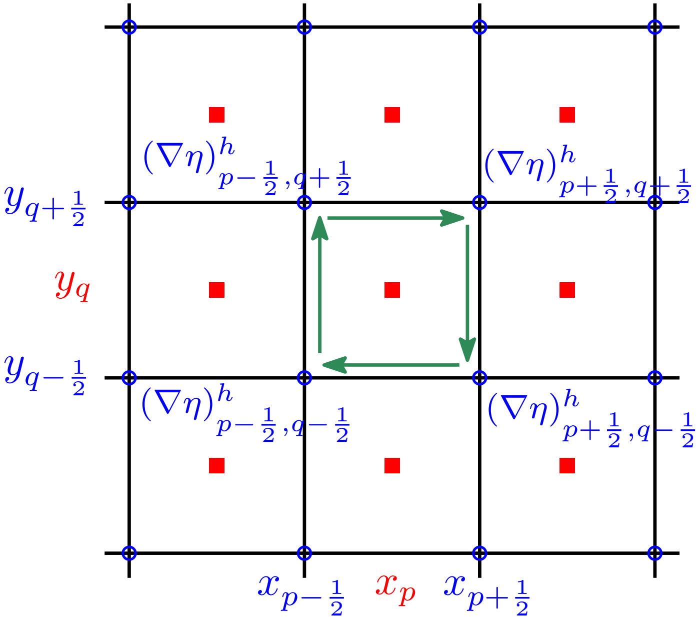


Figure 4: Stencil of the curl operator computed in every cell-center

Compatible discrete curl-operator

Based on this corner gradient, one can now define a compatible discrete curl operator such that $(\nabla^h \times \nabla^h \phi)^{p,q} \cdot \mathbf{e_z}$ is given by

$$\frac{(\partial_y^h \phi)^{p+\frac{1}{2},q+\frac{1}{2}} - (\partial_y^h \phi)^{p+\frac{1}{2},q-\frac{1}{2}}}{2\Delta x} + \frac{(\partial_y^h \phi)^{p-\frac{1}{2},q+\frac{1}{2}} - (\partial_y^h \phi)^{p-\frac{1}{2},q-\frac{1}{2}}}{2\Delta x}$$
$$-\frac{(\partial_x^h \phi)^{p+\frac{1}{2},q+\frac{1}{2}} - (\partial_x^h \phi)^{p-\frac{1}{2},q+\frac{1}{2}}}{2\Delta y} - \frac{(\partial_x^h \phi)^{p+\frac{1}{2},q-\frac{1}{2}} - (\partial_x^h \phi)^{p-\frac{1}{2},q-\frac{1}{2}}}{2\Delta y}.$$

It is straightforward to prove that

$$\nabla^h \times \nabla^h \phi \equiv 0$$

Update formulas ($h = \min(\Delta x, \Delta y)$)

- For the conserved variables $\rho, \mathbf{u}, \rho\eta, \rho w$:
  $\Rightarrow$ Classical MUSCL-Hancock scheme.

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- For the conserved variables $\rho, \mathbf{u}, \rho\eta, \rho w$:
  $\Rightarrow$ Classical MUSCL-Hancock scheme.
- For the curl-free vector $\mathbf{p}$

$$p_k^{p+\frac{1}{2},q+\frac{1}{2},n+1} = p_k^{p+\frac{1}{2},q+\frac{1}{2},n} - \Delta t\, \nabla_k^h \left(p_j u_j - w\right)^{p+\frac{1}{2},q+\frac{1}{2},n}$$

Update formulas ($h = \min(\Delta x, \Delta y)$)

- For the conserved variables $\rho, \mathbf{u}, \rho\eta, \rho w$:
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$$p_k^{p+\frac{1}{2},q+\frac{1}{2},n+1} = p_k^{p+\frac{1}{2},q+\frac{1}{2},n} - \Delta t \, \nabla_k^h \left(p_j u_j - w - h\, c^* \nabla_j^h p_j\right)^{p+\frac{1}{2},q+\frac{1}{2},n}$$

Update formulas ($h = \min(\Delta x, \Delta y)$)

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$$p_k^{p+\frac{1}{2},q+\frac{1}{2},n+1} = p_k^{p+\frac{1}{2},q+\frac{1}{2},n} - \Delta t\, \nabla_k^h \left(p_j u_j - w - h\ c^* \nabla_j^h p_j \right)^{p+\frac{1}{2},q+\frac{1}{2},n}$$

- Lastly, for $\mathbf{A}$

$$\begin{aligned} A_{ik}^{p+\frac{1}{2},q+\frac{1}{2},n+1} = {} & A_{ik}^{p+\frac{1}{2},q+\frac{1}{2},n} - \Delta t(\nabla_k^h(A_{ij}u_j) - h\ c^* \nabla_j^h A_{ij})^{p+\frac{1}{2},q+\frac{1}{2}} \\ & - \Delta t\ h\ c^* \varepsilon_{kj3} \nabla_j^{p+\frac{1}{2},q+\frac{1}{2},n} \left(\varepsilon_{3lm} \nabla_l^h A_{im} \right) \\ & - \frac{\Delta t}{4} \sum_{r=0}^{1} \sum_{s=0}^{1} u_m^{p+r,q+s,n} \left((\nabla_m^h A_{ik})^{p+\frac{1}{2},q+\frac{1}{2}} - (\nabla_k^h A_{im})^{p+\frac{1}{2},q+\frac{1}{2}} \right) \\ & - \Delta t \frac{1}{3\tau} \det(\mathbf{A}^{p+\frac{1}{2},q+\frac{1}{2},n+1})^{5/3} A_{im}^{p+\frac{1}{2},q+\frac{1}{2},n+1} \mathring{G}_{mk}^{p+\frac{1}{2},q+\frac{1}{2},n+1}. \end{aligned}$$

Near equilibrium bubble: density field

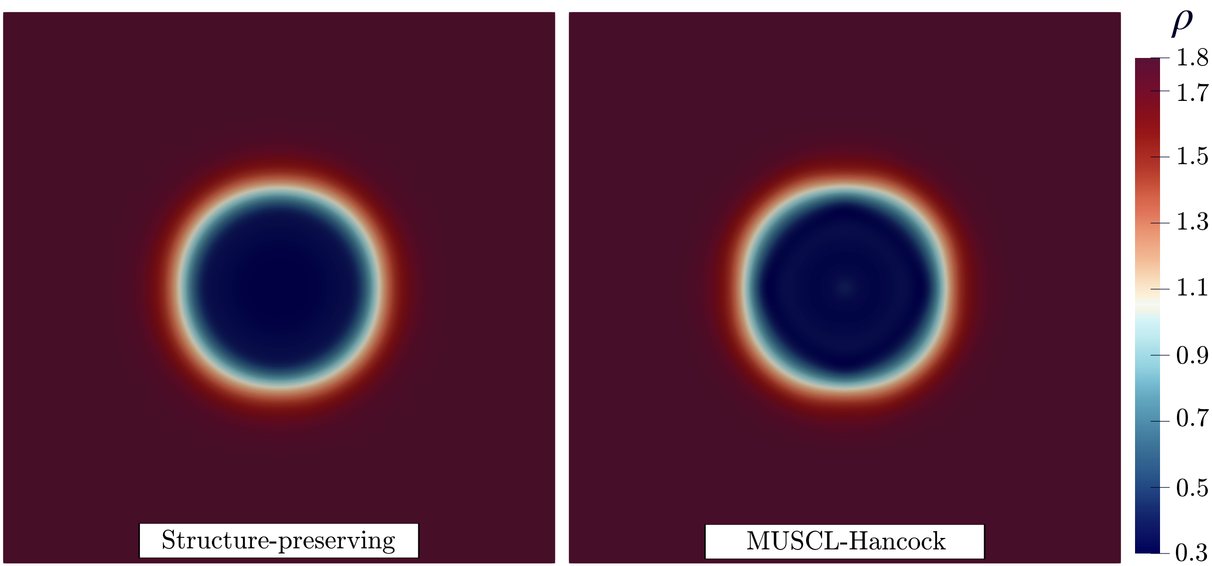

Figure 5: Results are shown for $t = 2$ on a 512×512 grid. With $\gamma = 2.10^{-4}$, $\alpha = 10^{-2}$, $\beta = 10^{-5}$, $\mu = 10^{-2}, c_s = 10$. The computational domain is $\Omega_c = [-0.25, 0.25] \times [-0.25, 0.25]$.

Near equilibrium bubble: gradient field

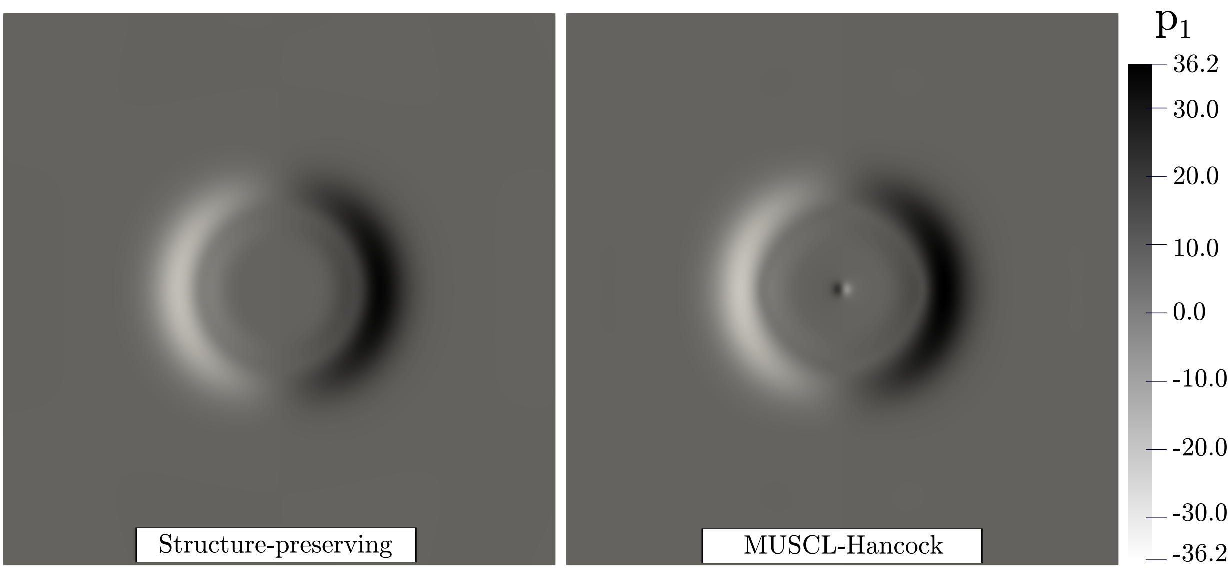

Figure 6: Results are shown for $t = 2$ on a 512×512 grid. With $\gamma = 2.10^{-4}$, $\alpha = 10^{-2}$, $\beta = 10^{-5}$, $\mu = 10^{-2}, c_s = 10$. The computational domain is $\Omega_c = [-0.25, 0.25] \times [-0.25, 0.25]$.

Near equilibrium bubble: Discrete curl error over time

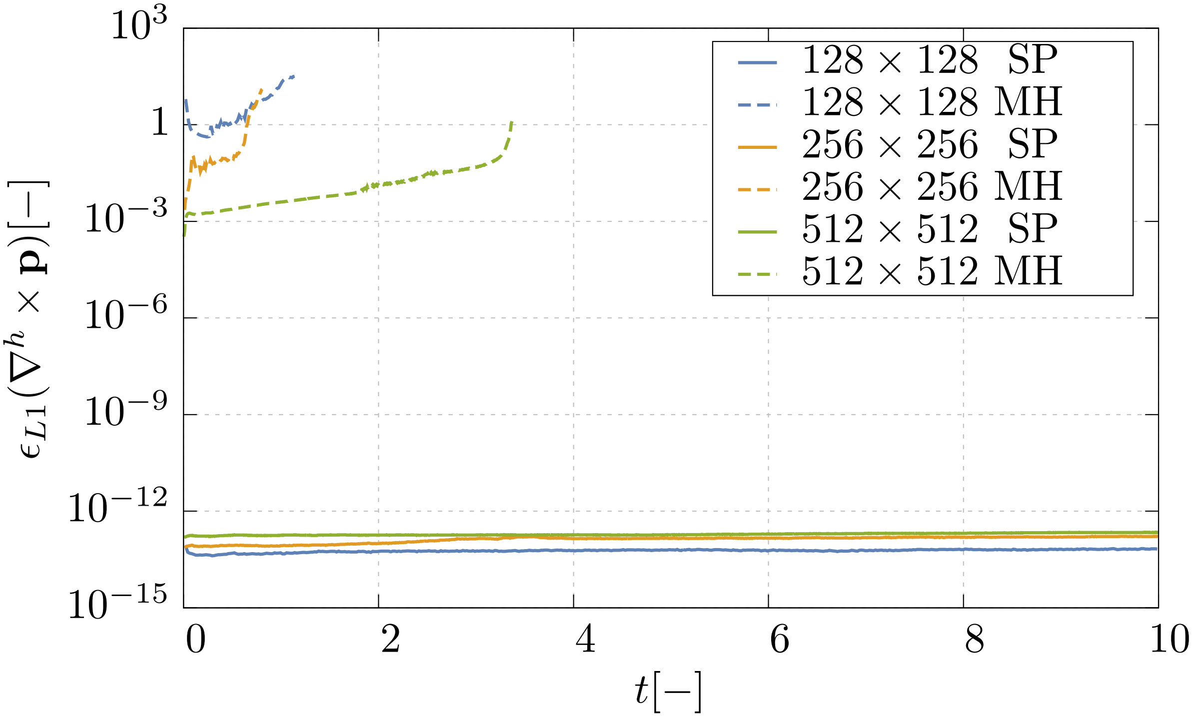

Figure 7: Time-evolution of the L_1 norm of the discrete curl errors on different mesh sizes.

2D Ostwald Ripening

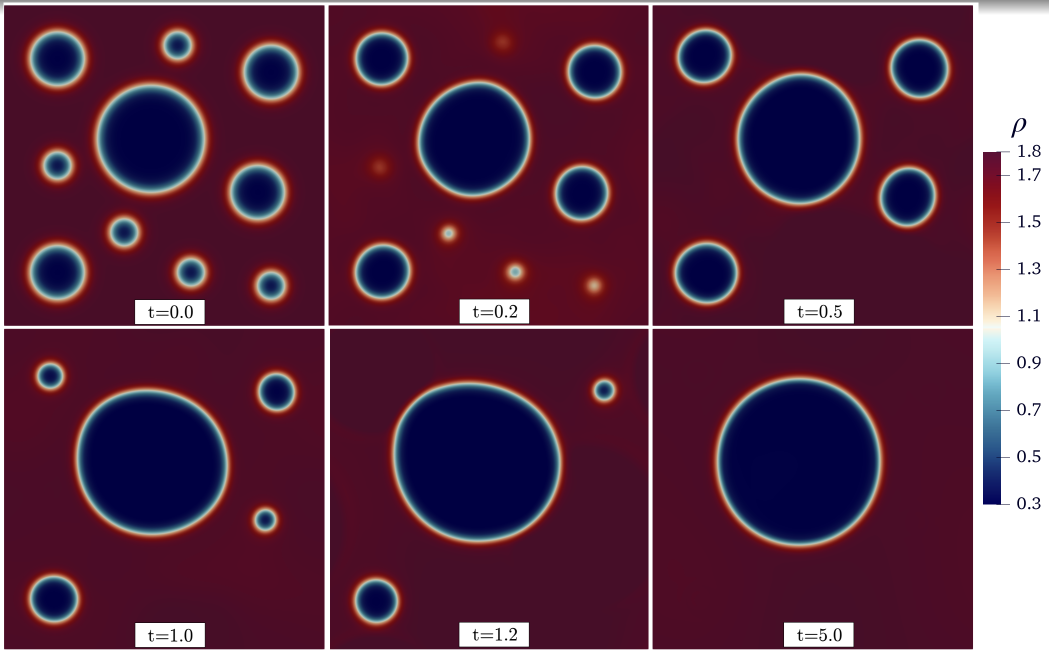

Figure 8: Values used here are $\rho_l = 1.8$, $\rho_v = 0.3$, $\gamma = 2.10^{-4}$, $\alpha = 10^{-2}$, $\beta = 10^{-5}$, $c_s = 10$ and an effective viscosity of $\mu = 10^{-2}$. The total domain is $\Omega = [-0.6, +0.6] \times [-0.6, +0.6]$ discretized over a 4096×4096 uniform grid with periodic boundary conditions.

Conclusion and Perspectives

Summary

- We conceived a hyperbolic relaxation model to the Navier-Stokes-Korteweg equations.
- The used scheme preserves the curl errors up to machine precision.
- Some numerical results blow up in finite time if a curl-free discretization is not used

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Summary

- We conceived a hyperbolic relaxation model to the Navier-Stokes-Korteweg equations.
- The used scheme preserves the curl errors up to machine precision.
- Some numerical results blow up in finite time if a curl-free discretization is not used

Perspectives

- Extension to non-isothermal flows.
- Splitting of the fluxes for semi-implicit discretization
- Higher-order extension of the scheme
- Thermodynamically compatible curl-free discretizations.

Thank you for your attention !

[1] Dhaouadi, Firas, and Michael Dumbser. "A first order hyperbolic reformulation of the Navier-Stokes-Korteweg system based on the GPR model and an augmented Lagrangian approach." *Journal of Computational Physics* 470 (2022): 111544.

[2] Dhaouadi, Firas, and Michael Dumbser. "A Structure-Preserving Finite Volume Scheme for a Hyperbolic Reformulation of the Navier–Stokes–Korteweg Equations." *Mathematics* 11.4 (2023): 876.

And references therein.

Dispersion relation

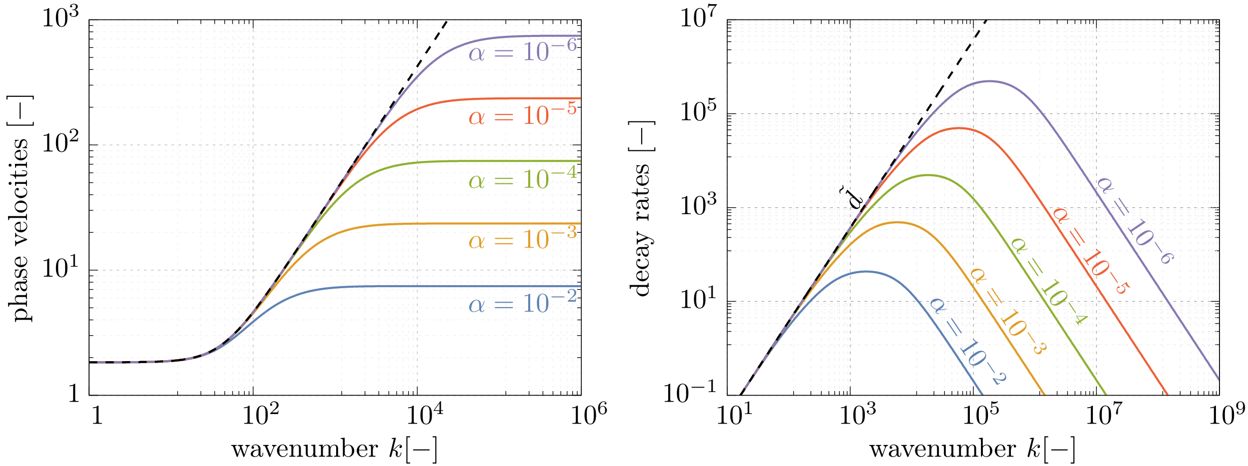

Figure 9: Plot of the phase velocity (left) and the decay rate for several values of α along their counterparts for the Navier-Stokes-Korteweg system. The model parameters are as follows $\gamma = 10^{-3}$, $\mu = 10^{-3}$ and $\rho = 1.8$

Scaling of relaxations

Representative characteristic velocities

$$\begin{cases} \lambda_{15} = u_1 - \sqrt{Z_1 + Z_2} \\ \lambda_{16} = u_1 - \sqrt{Z_1 - Z_2} \\ \lambda_{17} = u_1 + \sqrt{Z_1 + Z_2} \\ \lambda_{18} = u_1 + \sqrt{Z_1 - Z_2} \end{cases}, \begin{cases} Z_1 = \frac{1}{2}(a_0^2 + a_s^2 + a_\gamma^2 + a_\alpha^2 + a_\beta^2), \\ Z_2 = \sqrt{Z_1^2 - a_\beta^2(a_0^2 + a_\alpha^2 + a_s^2)}, \\ a_0 = \sqrt{\rho W''(\rho)}, \quad a_s = \sqrt{\frac{4}{3}c_s^2} \\ a_\alpha = \frac{\eta}{\rho\sqrt{\alpha}}, \quad a_\beta = \sqrt{\frac{\gamma}{\beta\rho}}, \quad a_\gamma = \sqrt{\frac{\gamma}{\rho}p_1^2} \end{cases}$$

The different relaxation contributions scale as

$$a_\alpha^2 \sim \frac{1}{\alpha}, \quad a_\beta^2 \sim \frac{\gamma}{\beta\rho}, \quad a_s^2 \sim c_s^2$$

To keep the contributions at the same order of magnitude, we can take for example

$$\beta = \gamma\alpha, \quad c_s = \frac{1}{\sqrt{\alpha}}$$