A hyperbolic augmented model for thin film flows

Firas Dhaouadi
Jean-Paul Vila
Nicolas Favrie
Sergey Gavrilyuk

Aix-Marseille Université - Université Toulouse III

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Introduction : Euler's equation for compressible fluids

A Lagrangian :

$$\mathcal{L}(\rho, \mathbf{u}) = \int_{\Omega_t} \left(\frac{\rho \left|\mathbf{u}\right|^2}{2} - \rho e(\rho) \right) d\Omega_t$$

A Constraint :

$$\rho_t + \mathrm{div}(\rho \mathbf{u}) = \mathbf{0}$$

$\Longrightarrow$ The corresponding Euler-Lagrange equation:

$$(\rho \mathbf{u})_t + \mathrm{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + p(\rho)\right) = 0; \quad p(\rho) = \rho^2 e'(\rho)$$

Dispersive models in mechanics

1. Surface waves with surface tension [Nikolayev, Gavrilyuk, Gouin 2006] :

$$\mathcal{L}(\mathbf{u}, h, \nabla h) = \int_{\Omega_t} \left(\frac{\rho_0 h \left|\mathbf{u}\right|^2}{2} - \frac{\rho_0 g h^2}{2} - \sigma \frac{\left|\nabla h\right|^2}{2} \right) d\Omega_t$$

2. Shallow water equations described by Serre-Green-Naghdi equations [Salmon (1998)]:

$$\mathcal{L}(u, h, \dot{h}) = \int_{\Omega_t} \left(\frac{hu^2}{2} - \frac{gh^2}{2} + \frac{h\dot{h}^2}{6} \right) d\Omega_t$$

Euler-Korteweg type systems

$$\mathcal{L}(\mathbf{u}, \rho, \nabla\rho) = \int_{\Omega_t} \left(\frac{\rho\,|\mathbf{u}|^2}{2} - A(\rho) - K(\rho)\frac{|\nabla\rho|^2}{2} \right) d\Omega_t$$

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho\mathbf{u}) = 0 \\ \partial_t(\rho\mathbf{u}) + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\rho) = \rho\nabla\left(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\right) \end{cases}$$

$K(\rho) = \sigma$: constant capillarity

$$\partial_t(\rho\mathbf{u}) + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\rho) = \sigma\rho\nabla\left(\Delta\rho\right)$$

$K(\rho) = \frac{1}{4\rho}$: Quantum capillarity / NLS equation (Shark 2018)

$$\partial_t(\rho\mathbf{u}) + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u} + \frac{1}{4\rho}\nabla\rho\otimes\nabla\rho) + \nabla\left(\frac{\rho^2}{2} - \frac{1}{4}\Delta\rho\right) = 0$$

The goal

Consider the following equations [Richard, Ruyer-Quil, Vila 2016]:

$$h_t + (hu)_x = 0$$

$$(hu)_t + \left(hu^2 + \frac{h^2}{2F^2} cos\theta + \frac{2\lambda^2 h^5}{225} \right)_x = \frac{1}{\varepsilon Re} \left(\lambda h - \frac{3u}{h} \right) + \frac{\kappa}{F^2} hh_{xxx}$$

Main Question

Can this system be solved by means of hyperbolic equations ?

The goal

Consider the following equations [Richard, Ruyer-Quil, Vila 2016]:

$$h_t + (hu)_x = 0$$

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Main Question

Can this system be solved by means of hyperbolic equations ?

Abstract

Yes.

Outline

Equations for thin films flow

We want to create a first order hyperbolic system of equations which approximates :

$$h_t + (hu)_x = 0$$

$$(hu)_t + \left(hu^2 + \frac{h^2}{2F^2}\cos\theta + \frac{2\lambda^2 h^5}{225} + \frac{\kappa}{2F^2} h_x^2 - \frac{\kappa}{F^2} h h_{xx} \right)_x = \frac{1}{\varepsilon Re}\left(\lambda h - \frac{3u}{h}\right)$$

- h and u are respectively the nondimensional fluid depth and average velocity.
- $\varepsilon = \frac{h_0}{L} \ll 1$

The corresponding Lagrangian

For the non frictional part of the previous set of equations

$$\mathcal{L}(\mathbf{u}, h, \nabla h) = \int_{\Omega_t} \left(h\frac{|\mathbf{u}|^2}{2} - A(h) - \frac{\kappa}{F^2}\frac{|\nabla h|^2}{2} \right) d\Omega_t$$

where : $A(h) = \dfrac{cos\theta}{2F^2}h^2 + \dfrac{\lambda^2}{450}h^5$

Energy conservation law:

$$\frac{\partial E}{\partial t} + \mathrm{div}\left(E\mathbf{u} + \Pi\mathbf{u} - \frac{\kappa}{F^2}\dot{h}\nabla h\right) = 0 \quad ; \qquad \dot{h} = h_t + \mathbf{u}\cdot\nabla h$$

where

$$E = h\frac{|\mathbf{u}|^2}{2} + A(h) + \frac{\kappa}{F^2}\frac{|\nabla h|^2}{2}$$

Extended Lagrangian approach

The objective

Obtain a new Lagrangian whose Euler-Lagrange equations :

- are hyperbolic
- accurately approximate thin films equations in a certain limit

The idea

- Decouple ∇h from $\mathbf{u}$ and h, have it as an independent variable.

Extended Lagrangian approach

SW Lagrangian :

$$\mathcal{L}(\mathbf{u}, h, \nabla h) = \int_{\Omega_t} \left(h\frac{|\mathbf{u}|^2}{2} - A(h) - \frac{\kappa}{F^2}\frac{|\nabla h|^2}{2} \right) d\Omega_t$$

'Extended' Lagrangian approach [Favrie, Gavrilyuk, 2017]

$$\tilde{\mathcal{L}}(\mathbf{u}, h, \eta, \nabla\eta, \dot{\eta}) \qquad \mathbf{p} = \nabla\eta \qquad w = \dot{\eta}$$

$$\tilde{\mathcal{L}} = \int_{\Omega_t} \left(h\frac{|\mathbf{u}|^2}{2} - A(h) - \frac{\kappa}{F^2}\frac{|\mathbf{p}|^2}{2} - \frac{h}{2\alpha}\left(1 - \frac{\eta}{h}\right)^2 + \frac{\beta h}{2} w^2 \right) d\Omega_t$$

$\frac{h}{2\alpha}\left(1 - \frac{\eta}{h}\right)^2$: Penalty

$\frac{\beta h}{2}\dot{\eta}^2$: Regularizer

Extended system Euler-Lagrange equations

The extended Lagrangian :

$$\tilde{\mathcal{L}} = \int_{\Omega_t} \left(h\frac{|\mathbf{u}|^2}{2} - A(h) - \frac{\kappa}{F^2}\frac{|\mathbf{p}|^2}{2} + \beta\frac{h}{2}w^2 - \frac{h}{2\alpha}\left(1 - \frac{\eta}{h}\right)\right) d\Omega_t$$

The constraint :

$$h_t + \mathrm{div}(h\mathbf{u}) = 0$$

$\Longrightarrow$ We apply Hamilton's principle :

$$\boxed{a = \int_{t_0}^{t_1} \tilde{\mathcal{L}}\, dt \quad \Longrightarrow \quad \delta a = 0}$$

Types of variations

Two types of variations will be considered :

$$\tilde{\mathcal{L}}(\overbrace{\mathbf{u}, \rho, \underbrace{\dot{\eta}}^{I}, \eta, \nabla\eta}_{II}) \qquad\qquad \dot{\eta} = \eta_t + \mathbf{u}\cdot\nabla\eta$$

- Type I : Virtual displacement of the continuum:

$$\hat{\delta} h = -\mathrm{div}(h\delta\mathbf{x}) \qquad \hat{\delta}\mathbf{u} = \dot{\delta\mathbf{x}} - \nabla\mathbf{u}\cdot\delta\mathbf{x} \qquad \delta\dot{\eta} = \hat{\delta}\mathbf{u}\cdot\nabla\eta$$

- Type II : Variations with respect to η

$$\delta\nabla\eta = \nabla\delta\eta \qquad \delta\dot{\eta} = (\delta\eta)_t + \mathbf{u}\cdot\nabla\delta\eta$$

Extended system Euler-Lagrange Equations

- Type I : Virtual displacement of the continuum:

$$\boxed{(h\mathbf{u})_t + \operatorname{div}\left(h\mathbf{u}\otimes\mathbf{u} + \mathbf{P}\right) = 0}$$

with $\mathbf{P} = \left(hA'(h) - A(h) - \frac{\kappa}{2F^2}\left|\mathbf{p}\right|^2 + \frac{\eta}{\alpha}(1 - \frac{\eta}{h})\right)\mathbf{Id} + \frac{\kappa}{F^2}\mathbf{p}\otimes\mathbf{p}$

- Type II : Variations with respect to η:

$$\boxed{(hw)_t + \operatorname{div}\left(hw\mathbf{u} - \frac{\kappa}{\beta F^2}\mathbf{p}\right) = \frac{1}{\alpha\beta}\left(1 - \frac{\eta}{h}\right)}$$

Closure of the system

$$w = \dot{\eta} = \eta_t + \mathbf{u} \cdot \nabla \eta \quad \Longrightarrow \quad \boxed{(hw)_t + div(h\eta\mathbf{u}) = 0}$$

$$\begin{aligned} \nabla w &= \nabla(\eta_t + \mathbf{u} \cdot \nabla \eta) \\ &= (\nabla \eta)_t + \nabla(\mathbf{u} \cdot \nabla \eta) \\ \Longrightarrow \quad & (\nabla \eta)_t + \nabla(\mathbf{u} \cdot \nabla \eta - w) = 0 \\ \Longrightarrow \quad & \boxed{\mathbf{p}_t + \mathrm{div}((\mathbf{p} \cdot \mathbf{u} - w)\mathbf{Id}) = 0} \end{aligned}$$

The full extended system

$$\begin{cases} h_t + \operatorname{div}(h\mathbf{u}) = 0 \\ (h\mathbf{u})_t + \operatorname{div}\left(h\mathbf{u}\otimes\mathbf{u} + \mathbf{P}\right) = 0 \\ (h\eta)_t + \operatorname{div}(h\eta\mathbf{u}) = hw \\ (hw)_t + \operatorname{div}\left(hw\mathbf{u} - \frac{\kappa}{\beta F^2}\mathbf{p}\right) = \frac{1}{\alpha\beta}\left(1 - \frac{\eta}{h}\right) \\ \mathbf{p}_t + \operatorname{div}\left(\left(\mathbf{p}\cdot\mathbf{u} - w\right)\mathbf{Id}\right) = 0; \qquad \operatorname{curl}(\mathbf{p}) = 0 \end{cases}$$

$$\mathbf{P} = \left(hA'(h) - A(h) - \frac{\kappa}{2F^2}\left|\mathbf{p}\right|^2 + \frac{\eta}{\alpha}(1 - \frac{\eta}{h})\right)\mathbf{Id} + \frac{\kappa}{F^2}\mathbf{p}\otimes\mathbf{p}$$

- Closed system.
- What about hyperbolicity ?
- Values of α and β ?

One-dimensional case:

In 1-d, the system reduces to :

$$\begin{cases} h_t + (hu)_x = 0 \\ (hu)_t + \left(hu^2 + \frac{h^2}{2F^2}\cos\theta + 2\lambda^2\frac{h^5}{225} + \frac{\kappa}{2F^2}p^2 + \frac{\eta}{\alpha}\left(1 - \frac{\eta}{h}\right)\right)_x = 0 \\ \eta_t + u\eta_x = w \\ w_t + uw_x - \frac{\kappa}{\beta h F^2}p_x = \frac{1}{\alpha\beta h}\left(1 - \frac{\eta}{h}\right) \\ p_t + up_x + pu_x - w_x = 0 \end{cases}$$

Reminder: The original equations we approximate :

$$h_t + (hu)_x = 0$$

$$(hu)_t + \left(hu^2 + \frac{h^2}{2F^2}\cos\theta + \frac{2\lambda^2 h^5}{225} + \frac{\kappa}{2F^2}h_x^2 - \frac{\kappa}{F^2}hh_{xx}\right)_x = \frac{1}{\varepsilon Re}\left(\lambda h - \frac{3u}{h}\right)$$

Relaxation / Couplings

$$\underbrace{w_t + u w_x}_{\dot{w}} - \frac{\kappa}{\beta h F^2} p_x = \frac{1}{\alpha\beta h}\left(1 - \frac{\eta}{h}\right)$$

$$\Rightarrow \boxed{h - \eta = \alpha\beta h^2 \dot{w} - \alpha \frac{\kappa h}{F^2} p_x}$$

$$\Rightarrow \boxed{h_x - \eta_x = h_x - p = \alpha\beta \left(h^2 \dot{w}\right)_x - \alpha \frac{\kappa h}{F^2} p_{xx}}$$

$$\Rightarrow \boxed{\frac{\eta}{\alpha}\left(1 - \frac{\eta}{h}\right) = \beta h \dot{w} - \frac{\kappa}{F^2} p_x = -\frac{\kappa}{F^2}\eta_{xx} + \mathcal{O}(\beta)}$$

One-Dimensional case : Hyperbolicity

In order to study the hyperbolicity of this system, we write it in quasi-linear form :

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = \mathbf{0}$$

where:

$$\mathbf{U} = \left(\; h, u, w, p, \eta \;\right)^T$$

$$\mathbf{A}(\mathbf{U}) = \begin{pmatrix} u & h & 0 & 0 & 0 \\ 1 + \frac{\cos\theta}{F^2} + \frac{\eta^2}{\alpha h^3} & u & 0 & \frac{\kappa}{hF^2} & \frac{1}{\alpha h}\left(1 - \frac{2\eta}{h}\right) \\ 0 & 0 & u & -\frac{\kappa}{\beta hF^2} & 0 \\ 0 & p & -1 & u & 0 \\ 0 & 0 & 0 & 0 & u \end{pmatrix}$$

One-Dimensional case : Hyperbolicity

The eigenvalues ζ of the matrix **A** are :

$$\boxed{\zeta = u,\ \zeta = u \pm \sqrt{\frac{b \pm \sqrt{b^2 - 4c}}{2}}.}$$

$$b = \frac{\kappa}{\beta h F^2}\left(1 + \beta p^2\right) + \frac{\cos\theta}{F^2} h + \frac{2}{5} h^4 + \frac{\eta^2}{\alpha h^2} > 0$$

$$c = \frac{\kappa}{\beta F^2}\left(\frac{\cos\theta}{F^2} + \frac{2}{5} h^3 + \frac{\eta^2}{\alpha h^3}\right) > 0$$

$\Longrightarrow$ $\boxed{\text{the system is always hyperbolic}}$

Adding back the source terms

Finally :

$$\begin{cases} h_t + (hu)_x = 0 \\ (hu)_t + \left(hu^2 + p(h) + \dfrac{\kappa}{2F^2}p^2 + \dfrac{\eta}{\alpha}\left(1 - \dfrac{\eta}{h}\right)\right)_x = \dfrac{1}{\varepsilon Re}\left(\lambda h - \dfrac{3u}{h}\right) \\ \eta_t + u\eta_x = w \\ w_t + uw_x - \dfrac{\kappa}{\beta h F^2}p_x = \dfrac{1}{\alpha\beta h}\left(1 - \dfrac{\eta}{h}\right) \\ p_t + up_x + pu_x - w_x = 0 \end{cases}$$

What about α and β ?

Values of α and β

- Values have to be assigned : a criterion is needed.
- We can base this choice, for example, on the dispersion relation.

Dispersion relation

We consider the small monochromatic perturbation of a constant equilibrium state $\mathbf{U}_0 =$ defined by :

$$\mathbf{U}(x,t) = \mathbf{U_0} + \mathbf{U}' e^{i(kx-\omega t)}$$

Then we look for $c_p = \frac{\omega}{k}$

Dispersion relation for the original system

We linearize the system around a constant state $(h_0, u_0 = \lambda h_0^2/3)$
$\Rightarrow$ The phase velocity satisfies :

$$(u_0 - c_p)^2 - ib(u_0 - c_p) - (a + 2ibu_0) = 0$$

with $a = \dfrac{cos\theta h_0}{F^2} + \dfrac{2\lambda^2 h_0^4}{45} + \kappa \dfrac{k^2}{F^2}; \qquad b = \dfrac{3}{k\varepsilon Reh_0^2}$ We want :

1. values of cp.
2. neutral stability condition.

Dispersion relation for the original system

Instead of calculating exact values of cp, we make expansions in a power series of the parameter ε :

$$cp = cp_0 + \varepsilon cp_1 + \mathcal{O}(\varepsilon^2)$$

$$cp' = \frac{1}{\varepsilon} cp'_{-1} + cp'_0 + \varepsilon cp'_1 \mathcal{O}(\varepsilon^2)$$

For the original system this gives :

$$cp = 3u_0 - \frac{iReh_0^2}{3}((\frac{cos\theta}{F^2} + \frac{5\lambda^2 h_0^3}{K_0} + \kappa \frac{k^2}{F^2})h_0 - 4u_0^2)k\varepsilon + \mathcal{O}(\varepsilon^2)$$

$$cp' = \frac{-3iu_0}{Reh_0^3 k\varepsilon} - u_0 - \frac{iReh_0^2}{3}((\frac{cos\theta}{F^2} + \frac{5\lambda^2 h_0^3}{K_0} + \kappa \frac{k^2}{F^2})h_0 - 4u_0^2)k\varepsilon + \mathcal{O}(\varepsilon^2$$

Neutral stability analysis

Since we are considering solutions which are proportional to $e^{i(kx-\omega t)}$, a necessary stability condition is :

$$Im(\omega) < 0 \Longleftrightarrow Im(c_p) < 0$$

Putting this inequality in the charactersitic polynomial gives (after some terrible calculations) :

$$\boxed{cotg\theta + \frac{\kappa k^2}{sin\theta} > \frac{6Re}{5}}$$

Neutral stability curves

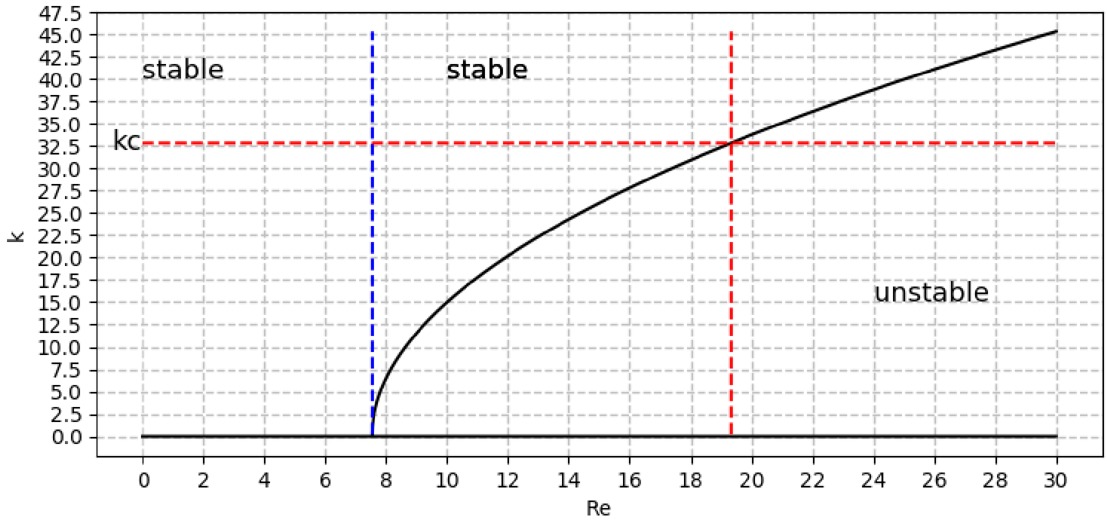


Figure 1: Plot of the critical stability curve ($\theta = 6.4^{\circ}$)

Liu & Gollub's experiment (1994)

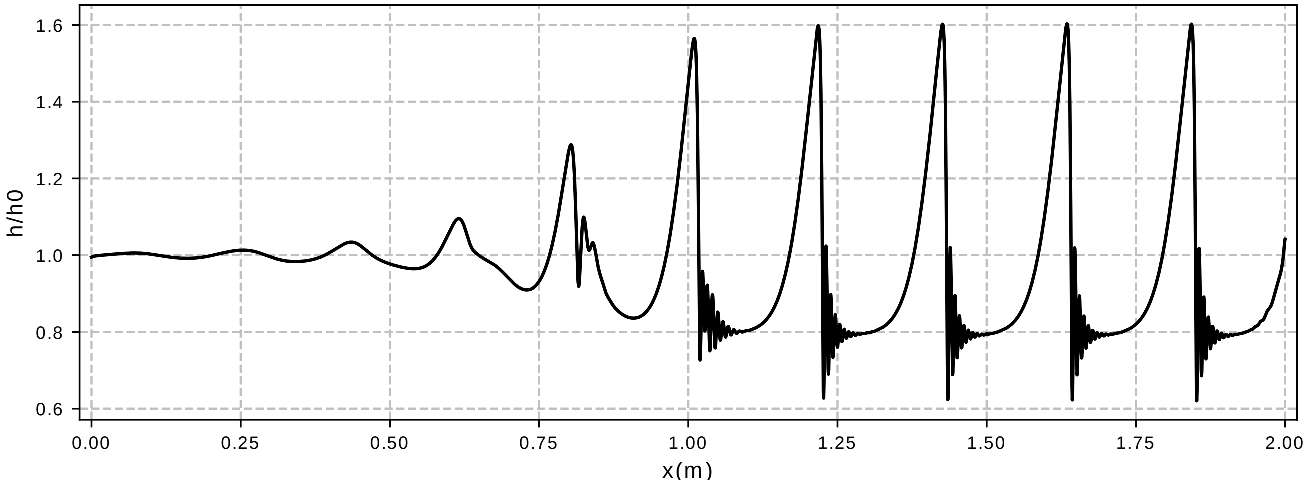


Figure 2: The Liu-Gollub experiment. The curve is the dimensionless depth of the wave profile in a 2.0 meter long canal, obtained with a forcing frequency f = 1.5Hz, imposed at the left boundary.

Dispersion relation for the extended model

Back to α and β. The suggested choice should be such that

1. The neutral stability analysis is consistent for both models.
2. The phase velocities are consistent to the first order for example.

Therefore we pose : $\alpha = \varepsilon^m$ and $\beta = \varepsilon^p$,calculate the characteristic polynomial for the augmented system and make expansions of the phase velocities as previously.

Main result

Consistency to the first order : $m > 1$ and $p > 2$
or equivalently : $\alpha < \varepsilon$ and $\beta < \varepsilon^2$

Neutral stability consistency

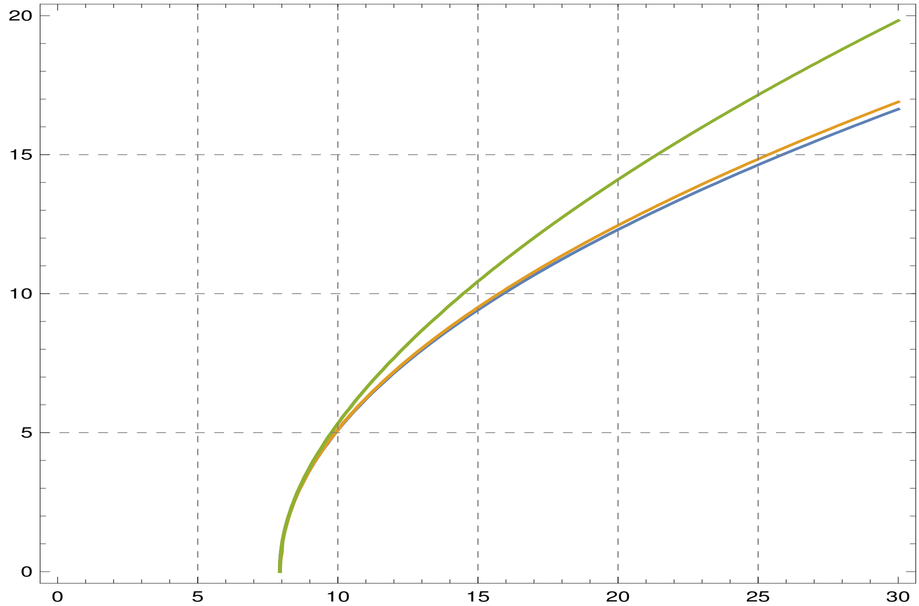


Figure 3: Comparison of both original(blue) and augmented model stability curves, or two sets of α, β.
orange : $\theta = 6.4°$, $\alpha = 0.01$,$\beta = 0.00003$
green: $\theta = 6.4°$, $\alpha = 0.1$,$\beta = 0.0001$

Numerical scheme: IMEX-Type

1-d system of equations to solve :

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S}(\mathbf{U})$$

The idea is to solve the hyperbolic part explicitly and the source term implicitly in time according to the scheme :

$$\mathbf{U}^{\star} = \mathbf{U}^{n} - \gamma \frac{\Delta t}{\Delta x}\left(F^{n}_{i+\frac{1}{2}} - F^{n}_{i-\frac{1}{2}}\right) + \gamma \Delta t \mathbf{S}(\mathbf{U}^{\star})$$

$$\mathbf{U}^{n+1} = \mathbf{U}^{n} - (\gamma - 1)\frac{\Delta t}{\Delta x}\left(F^{n}_{i+\frac{1}{2}} - F^{n}_{i-\frac{1}{2}}\right) - (2-\gamma)\frac{\Delta t}{\Delta x}\left(F^{\star}_{i+\frac{1}{2}} - F^{\star}_{i-\frac{1}{2}}\right) + (1-\gamma)\Delta t S(\mathbf{U}^{\star}) + \gamma \Delta t S(\mathbf{U}^{n+1})$$

Numerical scheme : Riemann solver

Riemann Solver: HLL-Rusanov.

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}\left(\mathbf{F}(\mathbf{U}^n_{i+1}) - \mathbf{F}(\mathbf{U}^n_i)\right) - \frac{1}{2}\kappa^n_{i+\frac{1}{2}}\left(\mathbf{U}^n_{i+1} - \mathbf{U}^n_i\right)$$

where $\kappa^n_{i+\frac{1}{2}}$ is obtained by using the Davis approximation :

$$\kappa^n_{i+1/2} = \max_j(|c_j(\mathbf{U}^n_i)|, |c_j(\mathbf{U}^n_{i+1})|),$$

where c_j are the eigenvalues of the extended system.

One last step before results

In order to compare with numerical results, viscosity must be added to the model :

$$(hu)_t + \left(hu^2 + \frac{h^2}{2F^2}cos\theta + \frac{2\lambda^2 h^5}{225}\right)_x = \frac{1}{\varepsilon Re}\left(\lambda h - \frac{3u}{h}\right) + \frac{\kappa}{F^2}hh_{xxx} + \frac{9\varepsilon}{2Re}(hu_x)_x$$

On the discrete level, we use centered finite differences :

$$(hu_x)_x = \left((hu_x)_{i+\frac{1}{2}} - (hu_x)_{i-\frac{1}{2}}\right)/\Delta x$$
$$(hu_x)_{i+\frac{1}{2}} = h_{i+\frac{1}{2}}\left(u_{i+1} - u_{i-1}\right)/\Delta x$$

Numerical result : f=1.5Hz

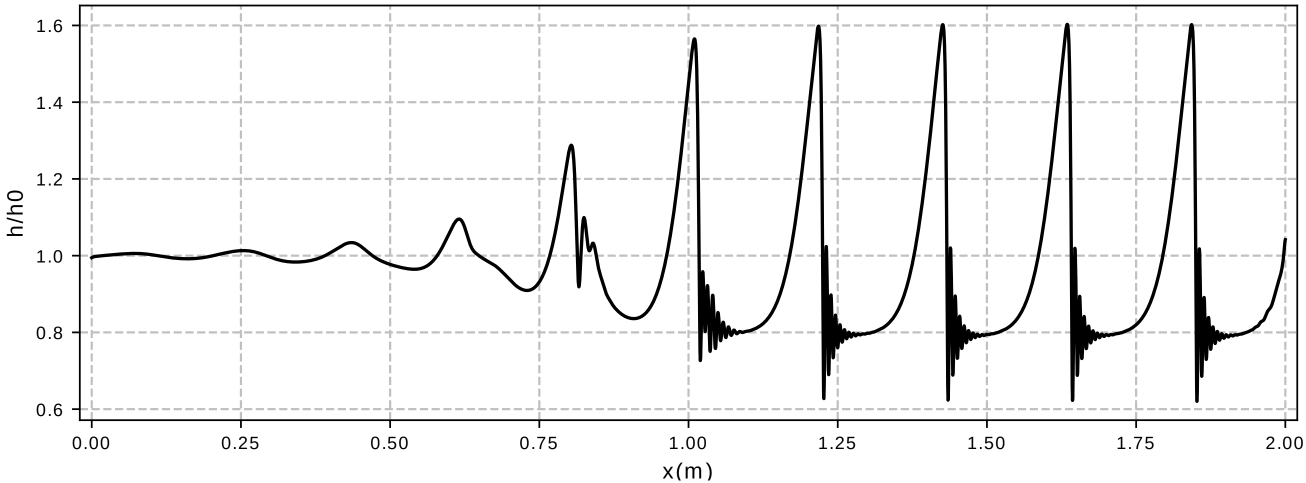


Figure 4: Numerical simulation of the Liu-Gollub experiment. The curve is the dimensionless depth of the wave profile in a 2.0 meter long canal, obtained with a forcing frequency f = 1.5Hz, imposed at the left boundary.

Numerical result : f=1.5Hz

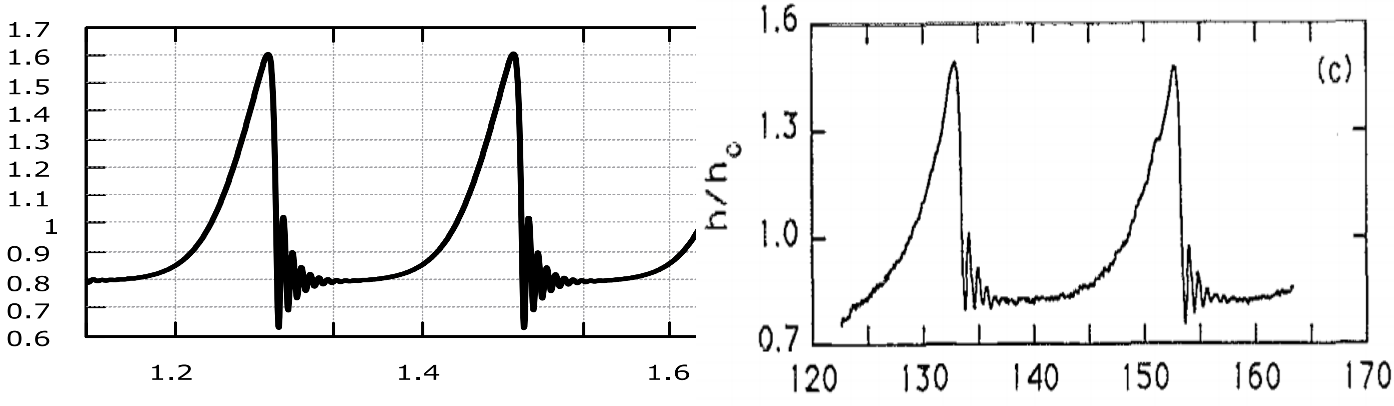


Figure 5: Comparison of the numerical simulation of the Liu-Gollub experiment with experimental data for f=1.5Hz ($\alpha = 0.005$, $\beta = 0.00003$, $\varepsilon = 0.0067$, nx=4000) boundary.

Numerical result : f=3.0Hz

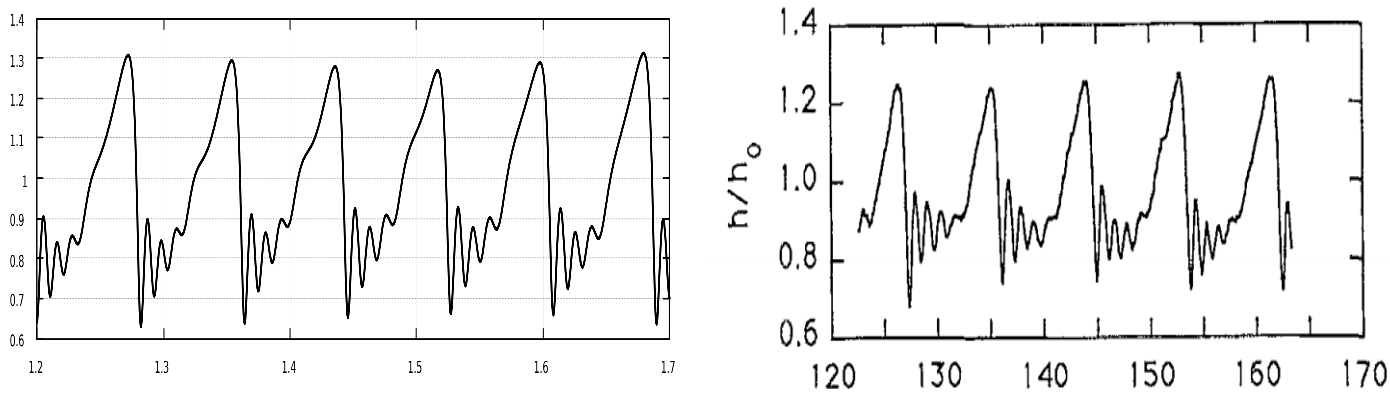
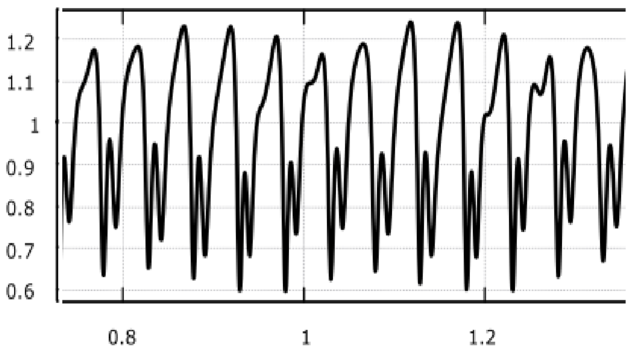


Figure 6: Comparison of the numerical simulation of the Liu-Gollub experiment with experimental data for f=3.0Hz ($\alpha = 0.005$, $\beta = 0.00003$, $\varepsilon = 0.0067$, nx=4000) boundary.

Numerical result : f=4.5Hz



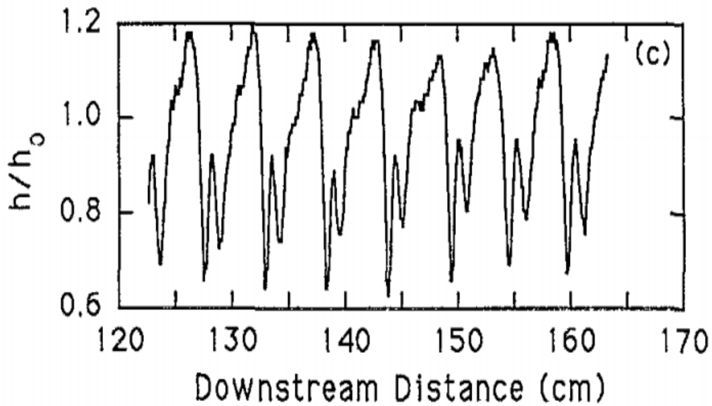


Figure 7: Comparison of the numerical simulation of the Liu-Gollub experiment with experimental data for f=4.5Hz ($\alpha = 0.005$, $\beta = 0.00003$, $\varepsilon = 0.0067$, nx=4000) boundary.

Conclusions - perspectives

Conclusions :

- We presented an approximate first order hyperbolic model for thin film flows based on an augmented Lagrangian method.
- The resulting system of equation is quite consistent with the original model.
- Preliminary tests look okay and are not expensive (5-6 times smaller timestepping)

Perspectives:

- Extension to the multidimensional case.
- Apply the same technique to the three equations model (with enstrophy)
- Further optimization of the numerical resolution.

Thank you for your attention :) !