A Hyperbolic reformulation of the Navier-Stokes-Korteweg equations

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Joint work with
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Navier-Stokes-Korteweg equations

In general, the equations write

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0 \\ (\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\rho) = \underline{\underline{S}} + \underline{\underline{K}} \end{cases}$$

where $\rho = \rho(\mathbf{x}, t)$, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and $(\mathbf{x}, t) \in \mathbb{R}^d \times [0, T]$
The (dispersive) Korteweg stress tensor is given by:

$$\underline{\underline{K}} = \rho\nabla\left(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\right)$$

and the (viscous) Navier-Stokes stresses are given by

$$\underline{\underline{S}} = \mu\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T - \frac{2}{3}\mathrm{div}(\mathbf{u})\mathbf{I}\right)$$

Dissipationless Euler-Korteweg equations

The equations write :

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \rho \nabla \left(K(\rho) \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2 \right) \end{cases}$$

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- $K(\rho) = \gamma$: **Compressible flow with surface tension**

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- $K(\rho) = \frac{1}{4\rho}$: **Quantum hydrodynamics**

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0 \\ (\rho\mathbf{u})_t + \operatorname{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + \frac{1}{4\rho}\nabla\rho\otimes\nabla\rho\right) + \nabla\left(\frac{\rho^2}{2} - \frac{1}{4}\Delta\rho\right) = 0 \end{cases}$$

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Surface tension / capillarity

- Euler-Korteweg equations : Fluid flow + Surface tension.
- Surface tension = Tendency of a fluid to shrink and minimize its surface.
- Examples in nature : Droplet shape, ripples on the water surface, water striders, etc...



Photos credits : pexels.com

Main objective

Given the Navier-Stokes-Korteweg system of equations :

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Suggested solution

A first-order hyperbolic reformulation of the NSK system!

More than just hyperbolic

We want a new model that:

- approximates Euler-Korteweg in some limit.
- is derived from a variational principle.
- admits no regions of ellipticity.
- is in line with the laws of thermodynamics.
- can be solved numerically with accurate numerical methods.

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Hyperbolic equations

- Mathematically well-posed equations.
- A very rich literature on numerical methods.
- Bounded wave speeds

Other Reformulations in a similar context

1. A family of Parabolic relaxation of NSK equations.

$\Rightarrow$ Rohde & collaborators [2014,2020]

$\Rightarrow$ Chertock *et al* [2017]

2. Hyperbolic approximation of Euler-Korteweg equations.
   - $\Rightarrow$ Dhaouadi *et al.*,2019. (Schrödinger equation)
   - $\Rightarrow$ Bourgeois *et al.* 2020 (Gradient solids with nonconvex EOS)
   - $\Rightarrow$ Bresch *et al.*,2020 (2nd Order Hyperbolic)
3. Hyperbolic reformulation of Navier-Stokes equations.
   - $\Rightarrow$ GPR model of continuum mechanics.[Godunov 1961,Romenski 1998,*Peshkov et al.* 2018]

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Idea

Combine the augmented Lagrangian model of Dhaouadi and the general Hyperbolic model of continuum mechanics of Godunov, Peshkov and Romenski.

Outline

1. Hyperbolic reformulation of Euler-Korteweg
2. Combination with the GPR model
3. Numerical results

Lagrangian for (EK) system

(EK) system can be derived from the Lagrangian :

$$\mathcal{L} = \int_{\Omega_t} \left(\frac{\rho\,|\mathbf{u}|^2}{2} - W(\rho) - K(\rho)\frac{|\nabla\rho|^2}{2} \right) \, d\Omega$$

Hamilton's principle

$+$

Differential constraint : $\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0$

$$(\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla(p(\rho)) = \rho\nabla\left(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2 \right)$$

with $p(\rho) = \rho W(\rho) - W(\rho)$

Augmented Lagrangian - Attempt 1

$$\mathcal{L}(\mathbf{u}, \rho, \nabla\rho) = \int_{\Omega_t} \left(\frac{1}{2}\rho\,|\mathbf{u}|^2 - W(\rho) - K(\rho)\frac{|\nabla\rho|^2}{2} \right) d\Omega$$

$$\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0$$

'Augmented' Lagrangian approach

$$\tilde{\mathcal{L}}(\mathbf{u}, \rho, \eta, \nabla\eta) \qquad (\eta \longrightarrow \rho)$$

$$\tilde{\mathcal{L}} = \int_{\Omega_t} \left(\rho\frac{|\mathbf{u}|^2}{2} - W(\rho) - K(\rho)\frac{|\nabla\eta|^2}{2} - \frac{1}{2\alpha\rho}(\rho - \eta)^2 \right) d\Omega$$

$$\frac{\rho}{2\alpha}\left(\frac{\eta}{\rho} - 1\right)^2$$: Classical Penalty term

Hints on calculus of variations

$$\tilde{\mathcal{L}} = \int_{\Omega_t} \left(\rho \frac{|\mathbf{u}|^2}{2} - W(\rho) - K(\rho) \frac{|\nabla \eta|^2}{2} - \frac{\rho}{2\alpha} \left(\frac{\eta}{\rho} - 1 \right)^2 \right) d\Omega$$

$$\tilde{\mathcal{L}}(\overbrace{\mathbf{u}, \rho}^{\delta \mathbf{x}}, \underbrace{\eta, \nabla \eta}_{\delta \eta}) \Rightarrow \text{Two Euler-Lagrange equations}$$

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- Virtual displacement of the continuum ($\delta \mathbf{x}$):

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla (\rho W'(\rho) - W(\rho))$$
$$= -\operatorname{div}(K(\rho) \nabla \eta \otimes \nabla \eta) - \nabla \left(\frac{1}{2} (\rho K'(\rho) - K(\rho)) |\nabla \eta|^2 + \frac{\eta}{\alpha} \left(1 - \frac{\eta}{\rho} \right) \right)$$

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- η variation ($\delta \eta$):

$$\frac{1}{\alpha} \left(1 - \frac{\eta}{\rho} \right) = -\left(K(\rho) \Delta \eta + K'(\rho) \nabla \rho \cdot \nabla \eta \right)$$

Preliminary system

Thus the system of governing equations now writes :

$$\begin{cases} \rho_t + \mathrm{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \mathrm{div}\left(\rho \mathbf{u} \otimes \mathbf{u}\right) + \nabla\left(\rho W'(\rho) - W(\rho)\right) = \mathrm{div}(\mathbf{K}_\alpha) \\ -K(\rho)\Delta\eta - K'(\rho)\nabla\rho \cdot \nabla\eta = \frac{1}{\alpha}\left(1 - \frac{\eta}{\rho}\right) \end{cases}$$

where:

$$\mathbf{K}_\alpha = -K(\rho)\nabla\eta \otimes \nabla\eta - \left(\frac{1}{2}(\rho K'(\rho) - K(\rho))|\nabla\eta|^2 + \frac{\eta}{\alpha}\left(1 - \frac{\eta}{\rho}\right)\right)\mathbf{Id}$$

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Original Korteweg stress tensor

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The obtained system :

✗ still contains high order derivatives.

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Idea : Include $\dot{\eta}$ into the Lagrangian !

Augmented Lagrangian - Attempt 2

Augmented Lagrangian approach

$$\tilde{\mathcal{L}}(\mathbf{u}, \rho, \eta, \nabla\eta, \dot{\eta}) \qquad \alpha, \beta \ll 1$$

$$\tilde{\mathcal{L}} = \int_{\Omega_t} \left(\rho \frac{|\mathbf{u}|^2}{2} - W(\rho) - K(\rho)\frac{|\nabla\eta|^2}{2} - \frac{\rho}{2\alpha}\left(\frac{\eta}{\rho} - 1\right)^2 + \frac{\beta\rho}{2}\dot{\eta}^2 \right) d\Omega$$

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Augmented Lagrangian approach

$$\tilde{\mathcal{L}}(\mathbf{u}, \rho, \eta, \nabla\eta, \dot{\eta}) \qquad \alpha, \beta \ll 1$$

$$\tilde{\mathcal{L}} = \int_{\Omega_t} \left(\rho \frac{|\mathbf{u}|^2}{2} - W(\rho) - K(\rho)\frac{|\nabla\eta|^2}{2} - \frac{\rho}{2\alpha}\left(\frac{\eta}{\rho} - 1\right)^2 + \frac{\beta\rho}{2}\dot{\eta}^2 \right) d\Omega$$

$\downarrow$ Hamilton's principle : $a = \int_{t_0}^{t_1} \tilde{\mathcal{L}}\, dt$

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0 \\ (\rho\mathbf{u})_t + \operatorname{div}\left(\rho\mathbf{u}\otimes\mathbf{u}\right) + \nabla\left(\rho W'(\rho) - W(\rho)\right) = \operatorname{div}(\mathbf{K}_\alpha) \\ (\beta\rho\dot{\eta})_t + \operatorname{div}\left(\beta\rho\dot{\eta}\mathbf{u} - K(\rho)\nabla\eta\right) = \frac{1}{\alpha}\left(1 - \frac{\eta}{\rho}\right) \end{cases}$$

Order reductions

1. We denote $w = \dot{\eta}$. Thus :

$$w = \eta_t + \mathbf{u} \cdot \nabla \eta \quad \Longrightarrow \quad \boxed{(\rho\eta)_t + \mathrm{div}(\rho\eta\mathbf{u}) = \rho w}$$

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Important : $\mathbf{p}(\mathbf{x}, t = 0) = \nabla \eta(\mathbf{x}, t = 0)$

Final form of the hyperbolic Euler-Korteweg system

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \operatorname{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + \left(\rho W'(\rho) - W(\rho)\right) \mathbf{Id} - \mathbf{K}_\alpha\right) = 0 \\ (\beta \rho w)_t + \operatorname{div}\left(\beta \rho w \mathbf{u} - K(\rho) \mathbf{p}\right) = \frac{1}{\alpha}\left(1 - \frac{\eta}{\rho}\right) \\ \mathbf{p}_t + \operatorname{div}((\mathbf{p} \cdot \mathbf{u} - w)\mathbf{Id}) = 0, \quad \operatorname{curl}(\mathbf{p}) = 0 \\ (\rho \eta)_t + \operatorname{div}(\rho \eta \mathbf{u}) = \rho w \end{cases}$$

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$\Rightarrow$ Strongly hyperbolic upgrade proposed in *Busto & al. 2021*.

Hyperbolicity in 1D, $K(\rho) = \gamma$

1D case: $\mathbf{u} = (u, 0, 0)^T$ and $\mathbf{p} = (p, 0, 0)^T$: We can write the system in its quasi-linear form

$$\mathbf{Q}_t + \mathbf{A}(\mathbf{Q})\mathbf{Q}_x = \mathbf{S}(\mathbf{Q})$$

where $\mathbf{Q}$ is the vector of primitive variables, $\mathbf{A} = \mathbf{A}(\mathbf{Q})$ is the jacobian matrix of the flux, and $\mathbf{S} = \mathbf{S}(\mathbf{Q})$ is the vector of source terms, all of which are given by

$$\mathbf{A} = \begin{pmatrix} u & \rho & 0 & 0 & 0 \\ a_{21} & u & 0 & \frac{\gamma p}{\rho} & a_{25} \\ 0 & 0 & u & -\frac{\gamma}{\beta\rho} & 0 \\ 0 & p & -1 & u & 0 \\ 0 & 0 & 0 & 0 & u \end{pmatrix}, \ \mathbf{Q} = \begin{pmatrix} \rho \\ u \\ w \\ p \\ \eta \end{pmatrix}, \ \mathbf{S} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\alpha\beta\rho}\left(1 - \frac{\eta}{\rho}\right) \\ 0 \\ w \end{pmatrix}$$

with $a_{21} = W''(\rho) + \frac{\eta^2}{\alpha\rho^3}$ and $a_{25} = \frac{1}{\alpha}\left(1 - \frac{2\eta}{\rho}\right)$

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$\mathbf{A}$ admits 5 eigenvalues that can be expressed as follows :

$$\xi = \begin{pmatrix} u \\ u + \sqrt{\psi_1 + \psi_2} \\ u + \sqrt{\psi_1 - \psi_2} \\ u - \sqrt{\psi_1 + \psi_2} \\ u - \sqrt{\psi_1 - \psi_2} \end{pmatrix} \text{ with } \begin{cases} \psi_1 = \frac{1}{2}(a^2 + a_\gamma^2 + a_\alpha^2 + a_\beta^2) \\ \psi_2 = \frac{1}{2}\sqrt{(a^2 + a_\gamma^2 + a_\alpha^2 - a_\beta^2)^2 + 4a_\beta^2 a_\gamma^2} \\ a = \sqrt{\rho W''(\rho)}, \quad a_\gamma = \sqrt{\frac{\gamma}{\rho} p^2} \\ a_\alpha = \frac{\eta}{\rho\sqrt{\alpha}}, \quad a_\beta = \sqrt{\frac{\gamma}{\beta\rho}} \end{cases}$$

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3. If $W''(\rho) < 0$, one can take α such that $a^2 + a_\alpha^2 > 0$.

$\Rightarrow$ Eigenvalues are always real for a reasonable choice of α.

Proof of hyperbolicity in 1D for $K(\rho) = \gamma$

Since $\psi_1 > 0$ and $\psi_2 \geq 0$, the eigenvalues are ordered as follows:

$$u - \sqrt{\psi_1 + \psi_2} \leq u - \sqrt{\psi_1 - \psi_2} < u < u + \sqrt{\psi_1 - \psi_2} \leq u + \sqrt{\psi_1 + \psi_2}$$

- Multiple eigenvalues for $\psi_2 = 0$.
- We can show that in this case, we still have a full basis of right eigenvectors:

$$\xi = \begin{pmatrix} u \\ u + \sqrt{\psi_1} \\ u + \sqrt{\psi_1} \\ u - \sqrt{\psi_1} \\ u - \sqrt{\psi_1} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -\frac{\rho - 2\eta}{\alpha a_\beta^2} & 0 & \frac{\rho}{a_\beta} & 0 & -\frac{\rho}{a_\beta} \\ 0 & 0 & 1 & 0 & 1 \\ 0 & -a_\beta & 0 & a_\beta & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This concludes the proof (works for general $K(\rho)$ [Dhaouadi 2020])

Some numerical results for hyperbolic EK equations

Preliminary test: The nonlinear Schrödinger equation

$$K(\rho) = \frac{1}{4\rho}, \quad W(\rho) = \rho^2/2$$

$$\begin{cases} \rho_t + \mathrm{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \mathrm{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + \left(\frac{\rho^2}{2} - \frac{1}{4}\Delta\rho\right)\mathbf{Id} + \frac{1}{4\rho}\nabla\rho \otimes \nabla\rho\right) = 0 \end{cases}$$

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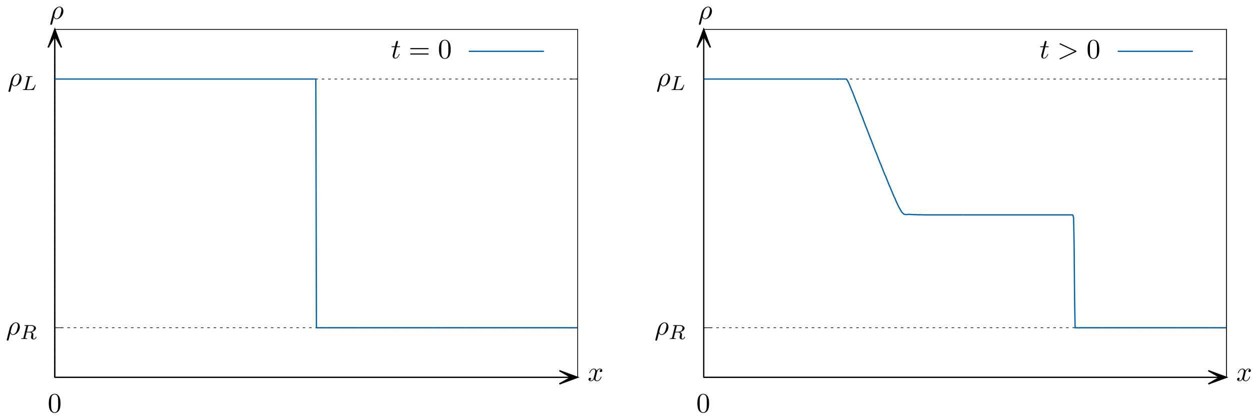
with

$$\psi(\mathbf{x}, t) = \sqrt{\rho(\mathbf{x}, t)} e^{i\theta(\mathbf{x}, t)} \qquad \mathbf{u} = \nabla\theta$$

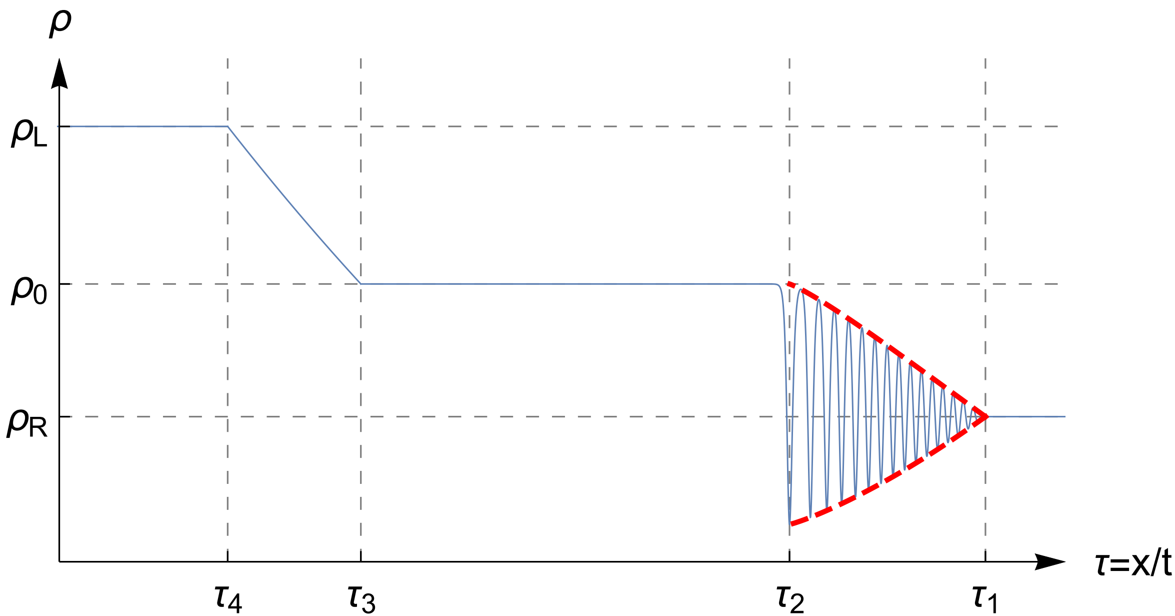
Shock waves for Euler equations

Riemann problem in dispersionless hydrodynamics governed by Euler Equations :



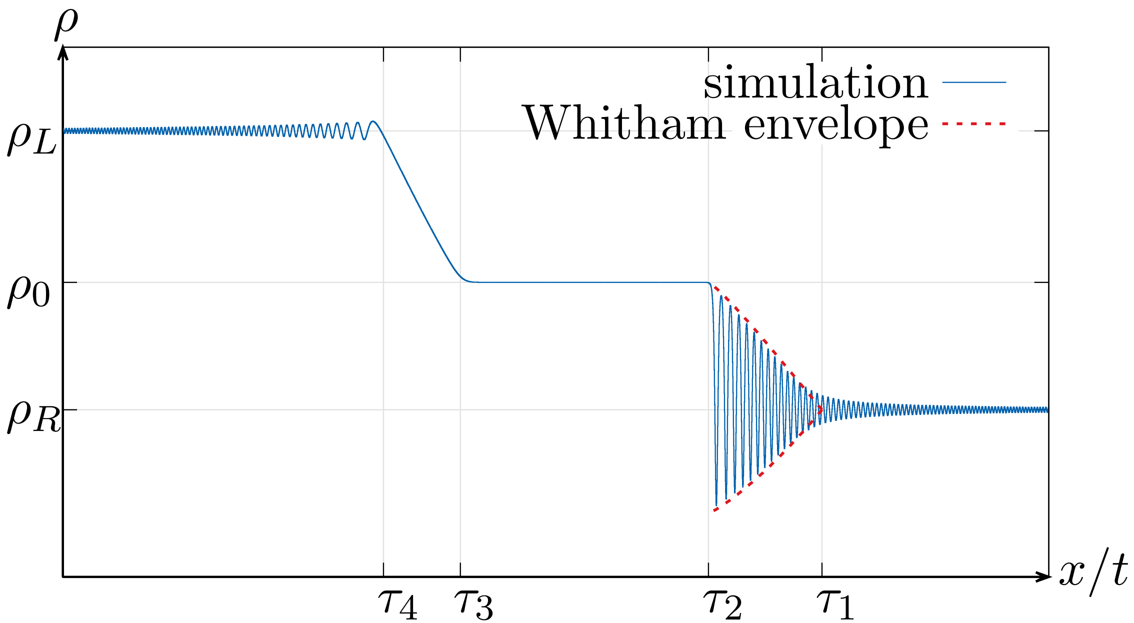
Rarefaction-Shock solution to a Riemann problem for Euler Equations.

Dispersive Shock waves



Asymptotic profile of the solution to NLS equation (continuous line) for the Riemann problem $\rho_L = 2$, $\rho_R = 1$, $u_L = u_R = 0$. Oscillations shown at t=70

DSW Numerical results



Comparison of the numerical result (ρ) with the Whitham modulational profile of the DSW at $t = 70$. $\beta = 2.10^{-5}$, $\alpha = 10^{-3}$, $N = 100000$. The computational domain is $[-500, 500]$

So far

- We proposed a first-order hyperbolic reformulation for the dispersive Euler-Korteweg equations.
- This reformulation remains hyperbolic even in non-convex regions of the free energy.
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- We proposed a first-order hyperbolic reformulation for the dispersive Euler-Korteweg equations.
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$\Rightarrow$ Let us extend this model to the Navier-Stokes-Korteweg system.

Navier-Stokes-Korteweg equations

In general, the equations write

$$\begin{cases} \rho_t + \mathrm{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \underline{\underline{S}} + \underline{\underline{K}} \end{cases}$$

where $\rho = \rho(\mathbf{x}, t)$, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and $(\mathbf{x}, t) \in \mathbb{R}^d \times [0, T]$
The (dispersive) Korteweg stress tensor is given by:

$$\underline{\underline{K}} = \rho \nabla \left(K(\rho) \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2 \right)$$

and the (viscous) Navier-Stokes stresses are given by

$$\underline{\underline{S}} = \mu \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3} \mathrm{div}(\mathbf{u}) \mathbf{I} \right)$$

Hyperbolic NSK = Hyperbolic EK + Hyperbolic NS

(Black: Euler part, Red: Dispersive part, Blue: Viscous part.)

$$
\begin{aligned}
&\partial_t(\rho) + \operatorname{div}(\rho\mathbf{u}) = 0 \\
&\partial_t(\rho\mathbf{u}) + \operatorname{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + (\rho W'(\rho) - W(\rho))\mathbf{Id} - K_\alpha - \sigma\right) = 0 \\
&\partial_t(\rho\eta) + \operatorname{div}(\rho\eta\mathbf{u}) = \rho w \\
&\partial_t(\rho w) + \operatorname{div}\left(\rho w\mathbf{u} - \frac{K(\rho)}{\beta}\mathbf{p}\right) = \frac{1}{\alpha\beta}\left(1 - \frac{\eta}{\rho}\right) \\
&\partial_t(\mathbf{p}) + \nabla\left(\mathbf{p}\cdot\mathbf{u} - w\right) = 0, \quad \mathbf{curl}(\mathbf{p}) = 0 \\
&\partial_t(\mathbf{A}) + \nabla(\mathbf{A}\mathbf{u}) + \left(\frac{\partial\mathbf{A}}{\partial\mathbf{x}} - \left(\frac{\partial\mathbf{A}}{\partial\mathbf{x}}\right)^T\right)\cdot\mathbf{u} = 0
\end{aligned}
$$

$$
\text{where}\begin{cases}\sigma = -\rho c_s^2\mathbf{G}\operatorname{dev}(\mathbf{G}) \text{ where } \mathbf{G} = \mathbf{A}^T\mathbf{A} \\ \mathbf{K}_\alpha = -\gamma\mathbf{p}\otimes\mathbf{p} + \left(\frac{\gamma}{2}|\mathbf{p}|^2 - \frac{\eta}{\alpha}\left(1 - \frac{\eta}{\rho}\right)\right)\mathbf{Id}\end{cases}
$$

GLM-curl cleaning approach

- By definition $\mathbf{p} = \nabla \eta$.

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In order to avoid spurious errors on $\mathbf{curl}(\mathbf{p})$ we use GLM-curl cleaning [Busto *et al.* 2021]:

$$\mathbf{p}_t - \nabla w + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T \mathbf{p} + \left(\frac{\partial \mathbf{p}}{\partial \mathbf{x}}\right) \mathbf{u} + 2a_c\rho\, \mathbf{curl}(\psi) = 0$$

$$\psi_t + \left(\frac{\partial \psi}{\partial \mathbf{x}}\right)^T \mathbf{u} - \frac{a_c}{2\rho}\mathbf{curl}(\mathbf{p}) = 0$$

with cleaning speed a_c that propagates $\mathbf{curl}(\mathbf{p})$ errors.

Reformulation of the NSK system

Black: Euler, Red: Dispersive, Blue: Viscous, Green: Curl Cleaning

$$\partial_t(\rho) + \mathrm{div}(\rho\mathbf{u}) = 0$$

$$\partial_t(\rho\mathbf{u}) + \mathrm{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + (\rho W'(\rho) - W(\rho))\mathbf{Id} - K_\alpha - \sigma\right) = 0$$

$$\partial_t(\rho\eta) + \mathrm{div}(\rho\eta\mathbf{u}) = \rho w$$

$$\partial_t(\rho w) + \mathrm{div}\left(\rho w\mathbf{u} - \frac{K(\rho)}{\beta}\mathbf{p}\right) = \frac{1}{\alpha\beta}\left(1 - \frac{\eta}{\rho}\right)$$

$$\mathbf{p}_t - \nabla w + \left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T\mathbf{p} + \left(\frac{\partial\mathbf{p}}{\partial\mathbf{x}}\right)\mathbf{u} + 2a_c\rho\,\mathbf{curl}(\psi) = 0$$

$$\psi_t + \left(\frac{\partial\psi}{\partial\mathbf{x}}\right)^T\mathbf{u} - \frac{a_c}{2\rho}\mathbf{curl}(\mathbf{p}) = 0$$

$$\partial_t(\mathbf{A}) + \nabla(\mathbf{A}\mathbf{u}) + \left(\frac{\partial\mathbf{A}}{\partial\mathbf{x}} - \left(\frac{\partial\mathbf{A}}{\partial\mathbf{x}}\right)^T\right)\cdot\mathbf{u} = 0$$

Van der Waals equation of state

In the context of two-phase flows, the equation of state is non-convex

$$p = \frac{\rho R T}{1 - b\rho} - a\rho^2, \qquad a > 0,\ b > 0$$

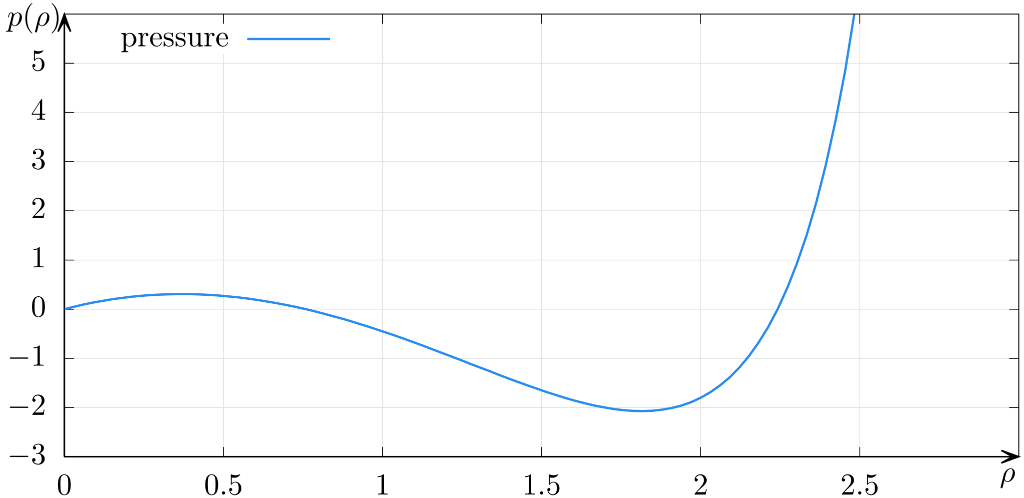


Figure 1: Van der Waals pressure for $T = 0.85, a = 3, b = 1/3, R = 8/3$

1D Traveling wave solutions for original NSK

1D NSK system reduces to:

$$\partial_t(\rho) + \partial_x(\rho u) = 0$$

$$\partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = \frac{4}{3}\mu u_{xx} + \gamma \rho \rho_{xxx}$$

Traveling wave assumption: $\rho(x,t) = \rho(x - st)$, $u(x,t) = u(x - st)$

$$\begin{cases} \rho''' = \dfrac{1}{\lambda\rho}\left(\left(p'(\rho) - (u-s)^2\right)\rho' - \dfrac{4}{3}\mu(u-s)(2\dfrac{\rho'^2}{\rho^2} - \dfrac{\rho''}{\rho})\right) \\ u' = (s-u)\dfrac{\rho'}{\rho} \end{cases}$$

which we solve as a Cauchy problem with a prescribed initial condition $\rho_0 = 1.8$, $\rho_0' = -10^{-10}$, $\rho_0'' = 0$, $u_0 = 0$

Traveling wave solutions

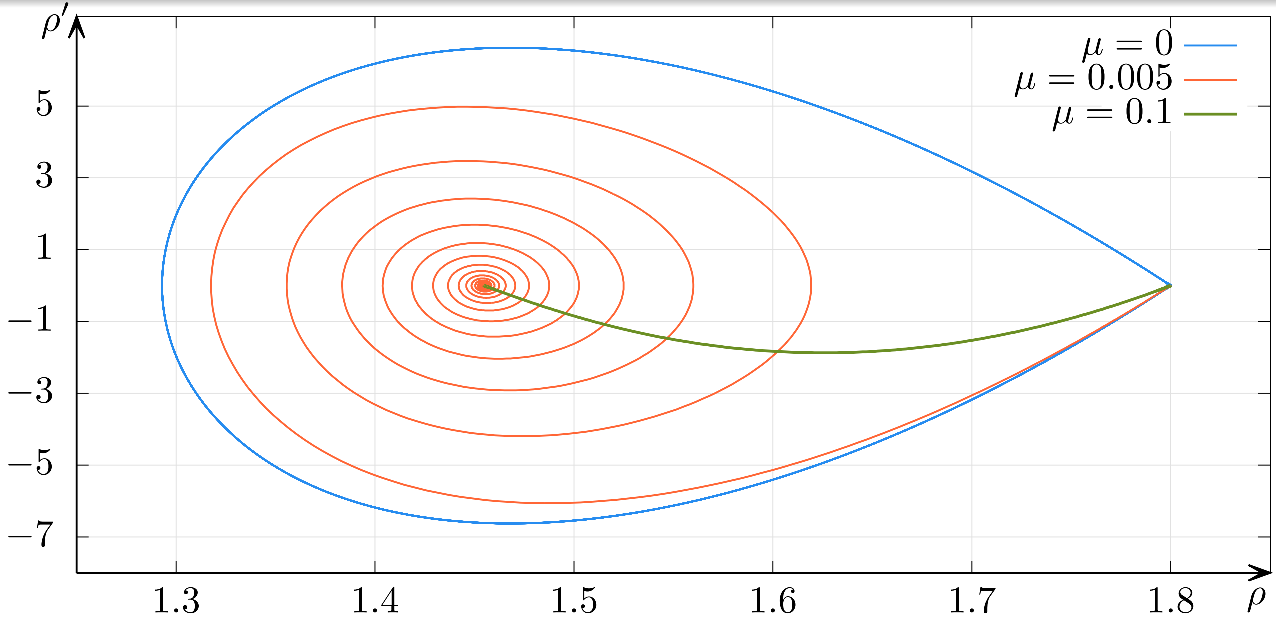
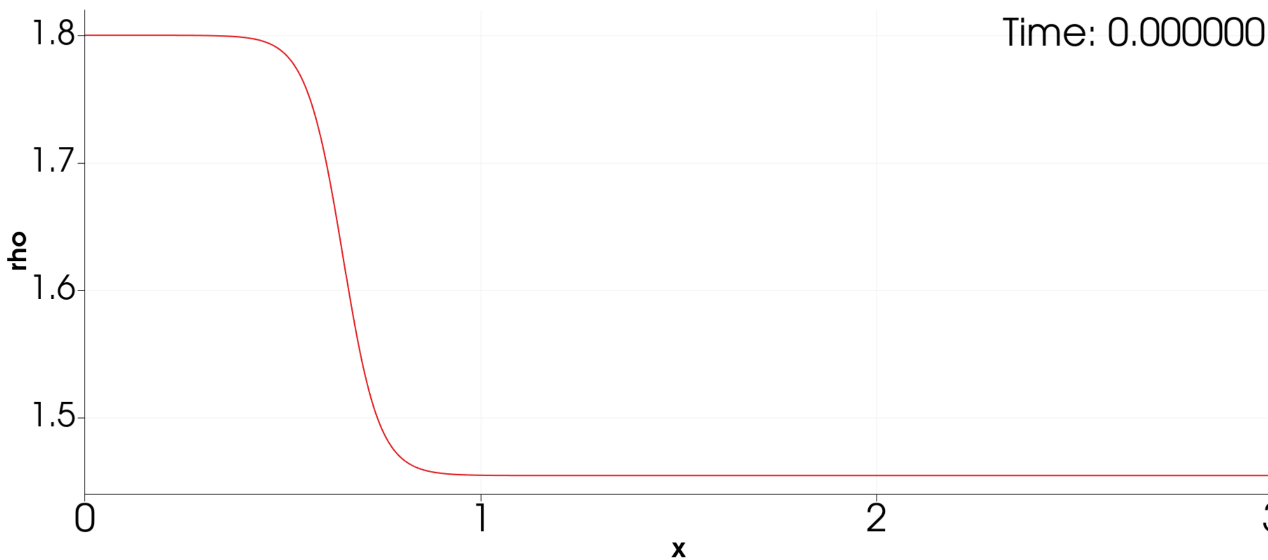


Figure 2: Nature of travelling wave solutions at fixed dispersion ($\gamma = 0.001$), for the original NSK equations.

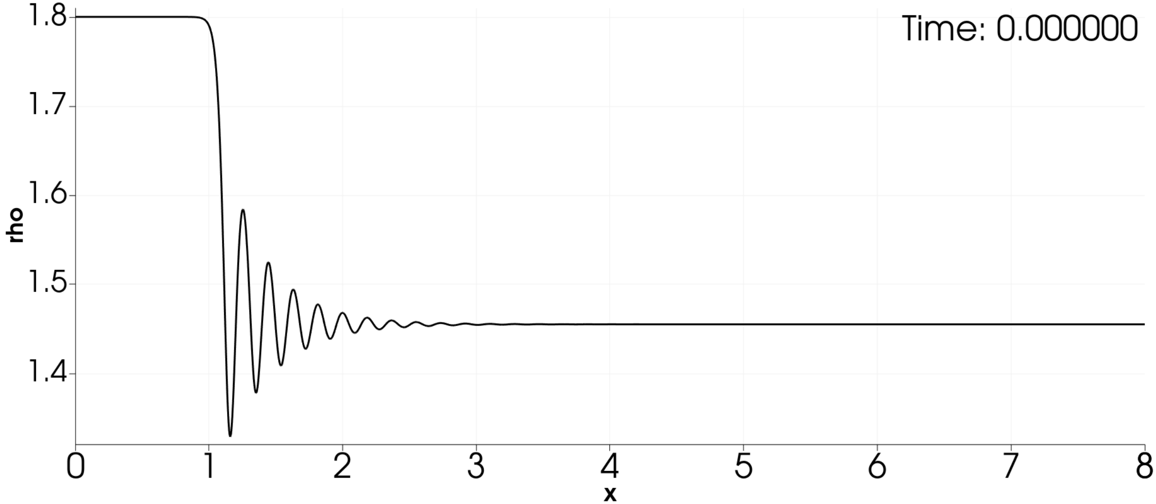
See [Affouf & Caflisch 1991] for a discussion on the nature of the solutions for a simplified system.

Viscous shock waves



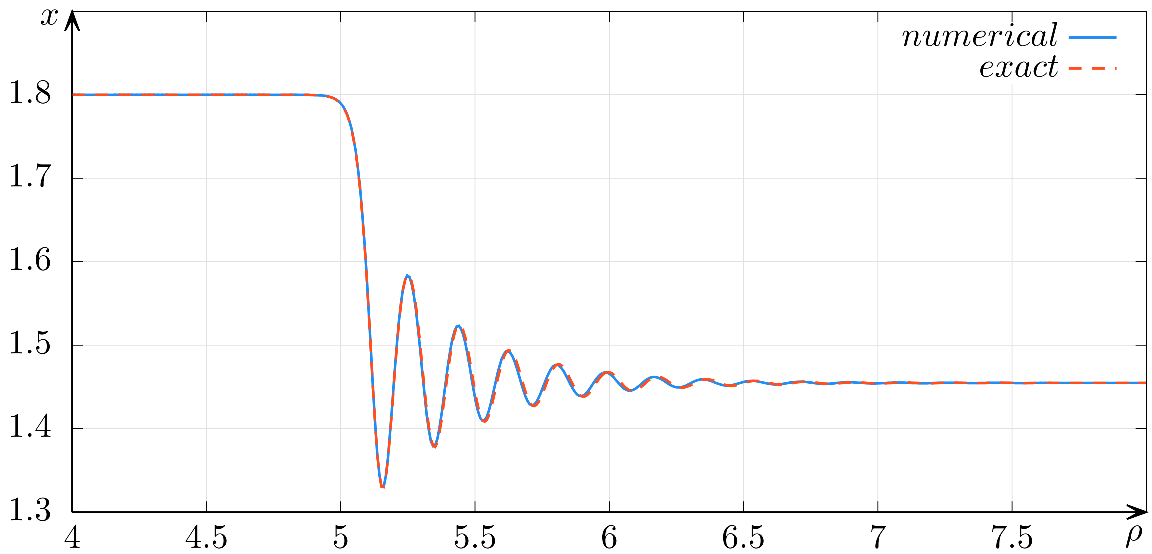
Viscous shock traveling wave solution to the original NSK (Obtained with a P_4P_4 ADER-DG scheme + WENO3 subcell limiting on a grid with 512 cells with $\gamma = 0.001$,$\mu = 0.2$, $\alpha = 0.001$, $\beta = 0.00001$)

Dispersive shock waves



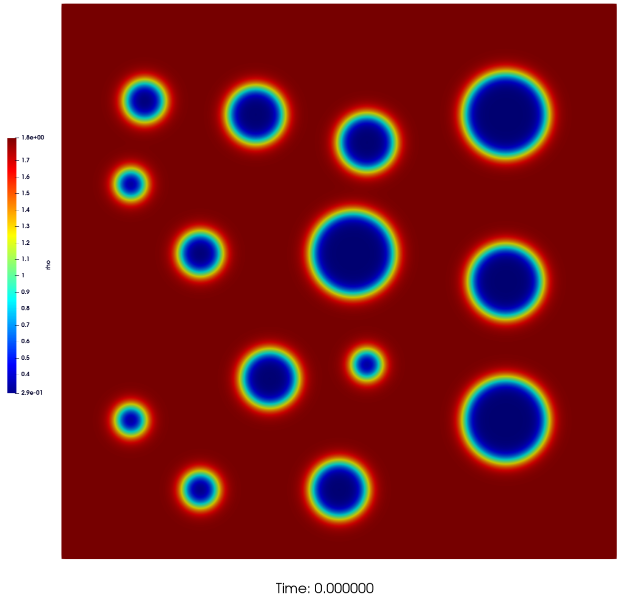
Dispersive shock traveling wave solution to the original NSK (Obtained with a P_4P_4 ADER-DG scheme + WENO3 subcell limiting on a grid with 512 cells with $\gamma = 0.001$,$\mu = 0.0075$, $\alpha = 0.001$, $\beta = 0.00001$)

Dispersive shock wave



Superimposed numerical solution and exact solution of original model at t=4. (Obtained with a P_4P_4 ADER-DG scheme + WENO3 subcell limiting on a grid with 512 cells with $\gamma = 0.001$,$\mu = 0.0075$, $\alpha = 0.001$, $\beta = 0.00001$)

2D Ostwald Ripening



Ostwald ripening test case with 14 bubbles (Obtained with a P_3P_3 ADER-DG scheme + Periodic boundary conditions + WENO3 subcell limiting on a 192×192 grid with$\gamma = 0.001$,$\mu = 0.01$, $\alpha = 0.001$, $\beta = 0.00001$)

Perspectives

- Further investigations from the physics' point of view (Young-Laplace jumps, phase change, contact angles).
- Application of structure preserving schemes, in particular exactly curl-free schemes.
- Splitting of the fluxes to separate fast waves for less constraining time-steps.
- Investigation of the sharp interface limit ($\gamma \to 0$) and asymptotic preserving schemes.
- Generalization of the hyperbolic model to the non-isothermal case.