A hyperbolic model for heat transfer in compressible flows.

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Abstract

We present a new model for heat transfer in compressible fluid flows. The model is derived from Hamilton's principle of stationary action in Eulerian coordinates, in a setting where the entropy conservation is recovered as an Euler–Lagrange equation. The governing system is shown to be hyperbolic. It is asymptotically consistent with the Euler equations for compressible heat conducting fluids, provided the addition of suitable relaxation terms. A study of the Rankine–Hugoniot conditions and the Clausius–Duhem inequality reveals that contact discontinuities cannot exist while expansion waves and compression fans are possible solutions to the governing equations. Evidence of these properties is provided on a set of numerical test cases.

1. Entropy as an Euler-Lagrange eqn

The idea is to introduce a scalar field ϕ such that the temperature is $\theta = \dot{\phi}$. In this setting one writes the Lagrangian

$$\mathcal{L} = \int_\Omega \left(\frac{1}{2}\rho \|\mathbf{u}\|^2 - \frac{\varkappa^2}{2\rho}\|\nabla\phi\|^2 - \rho\varepsilon^\star(\rho,\dot{\phi}) \right) d\Omega,$$

$\varepsilon^\star$ is the Legendre transform of internal energy and η is the entropy

$$\varepsilon(\rho,\eta) = \varepsilon^\star(\rho,\dot{\phi}) - \eta\dot{\phi}, \quad \text{with} \quad \eta = \frac{\partial\varepsilon^\star}{\partial\dot{\phi}}.$$

The Euler-Lagrange equation for ϕ is then

$$\frac{\partial}{\partial t}(\rho\eta) + \operatorname{div}(\rho\eta\mathbf{u} + \alpha(\rho)\nabla\phi) = 0.$$

2. Motion equations

One promotes the variable $\mathbf{j} = \nabla\phi$. Then, the system of equations is given by

$$\begin{aligned}
&\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = 0,\\
&\frac{\partial\rho\mathbf{u}}{\partial t} + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u} + \Pi) = 0,\\
&\frac{\partial\mathbf{j}}{\partial t} + \nabla(\mathbf{j}\cdot\mathbf{u} + \theta) + \left(\frac{\partial\mathbf{j}}{\partial\mathbf{x}} - \left(\frac{\partial\mathbf{j}}{\partial\mathbf{x}}\right)^T\right)\mathbf{u} = -\frac{\partial\mathcal{R}}{\partial\mathbf{j}},\\
&\frac{\partial\rho\eta}{\partial t} + \operatorname{div}\left(\rho\eta\mathbf{u} + \frac{\varkappa^2}{\rho}\mathbf{j}\right) = \frac{\varkappa^2}{\rho\theta}\frac{\partial\mathcal{R}}{\partial\mathbf{j}}\cdot\mathbf{j}.
\end{aligned}$$

where $\Pi = \left(\rho^2\frac{\partial\varepsilon}{\partial\rho} - \frac{\varkappa^2}{\rho}\|\mathbf{j}\|^2\right)\mathbf{I} + \frac{\varkappa^2}{\rho}\mathbf{j}\otimes\mathbf{j}$

$\mathcal{R} = \frac{1}{2\tau}\|\mathbf{j}\|^2$ is the Rayleigh dissipation function.

The system admits an energy conservation law

$$\frac{\partial E}{\partial t} + \operatorname{div}\left(E\mathbf{u} + \Pi\mathbf{u} + \frac{\varkappa^2\theta}{\rho}\mathbf{j}\right) = 0$$

3. Quick overview of the model

The following properties are provable

1. Energy is convex in the conservative variables.
2. The system can be symmetrized and is proven to be hyperbolic.
3. Fourier's law of heat conduction can be recovered in the relaxation limit $\tau \to 0$.

In $1d$, it admits the eigenvalues

$$\begin{cases}\lambda_1 = u - \sqrt{Y_1 + Y_2},\\ \lambda_2 = u - \sqrt{Y_1 - Y_2},\\ \lambda_3 = u + \sqrt{Y_1 - Y_2},\\ \lambda_4 = u + \sqrt{Y_1 + Y_2},\end{cases} \quad \text{where} \quad \begin{cases} Y_1 = \frac{1}{2}\left(a_p^2 + a_T^2\right),\\ Y_2 = \sqrt{a_{pT}^4 + Y_3^2},\\ Y_3 = \frac{1}{2}\left(a_p^2 - a_T^2\right).\end{cases}$$

where $a_p^2 = \dfrac{\partial p}{\partial\rho}$, $\quad a_T^2 = \dfrac{\varkappa^2}{\rho^2}\dfrac{\partial\theta}{\partial\eta}$, $\quad a_{pT}^4 = \dfrac{\varkappa^2}{\rho^2}\dfrac{\partial p}{\partial\eta}\dfrac{\partial\theta}{\partial\rho}$

The system is also hyperbolic in multi-dimensions if Godunov-Powell terms or curl-cleaning are also supplied.

4. Clausius-Duhem inequality

One can write the Rankine-Hugoniot conditions for our system

$$(\mathcal{M} = \rho(u - \mathcal{D})) \qquad [\mathcal{M}] = 0,$$

$$\left[p + \frac{\mathcal{M}^2}{\rho}\right] = 0,$$

$$\left[\mathcal{M}\left(\frac{\mathcal{M}^2}{2\rho^2} + \varepsilon + \frac{p}{\rho} + \frac{1}{2}\frac{\varkappa^2}{\rho^2}j^2\right) + \frac{\varkappa^2}{\rho}\theta\, j\right] = 0,$$

$$\left[\mathcal{M}\frac{j}{\rho} + \theta\right] = 0,$$

Through shocks, the jumps are supplemented with

$$\left[\rho(u - \mathcal{D})\eta + \frac{\varkappa^2}{\rho}j\right] \geq 0,$$

as a necessary condition for shock admissibility :

$$\Psi(v) = \eta(v) - \eta_0 - \frac{\varkappa^2}{\mathcal{M}(v)^2}(\theta(v) - \theta_0) \geq 0$$

if the subscript 0 is the state ahead of shock.

5. Existence of expansion shocks

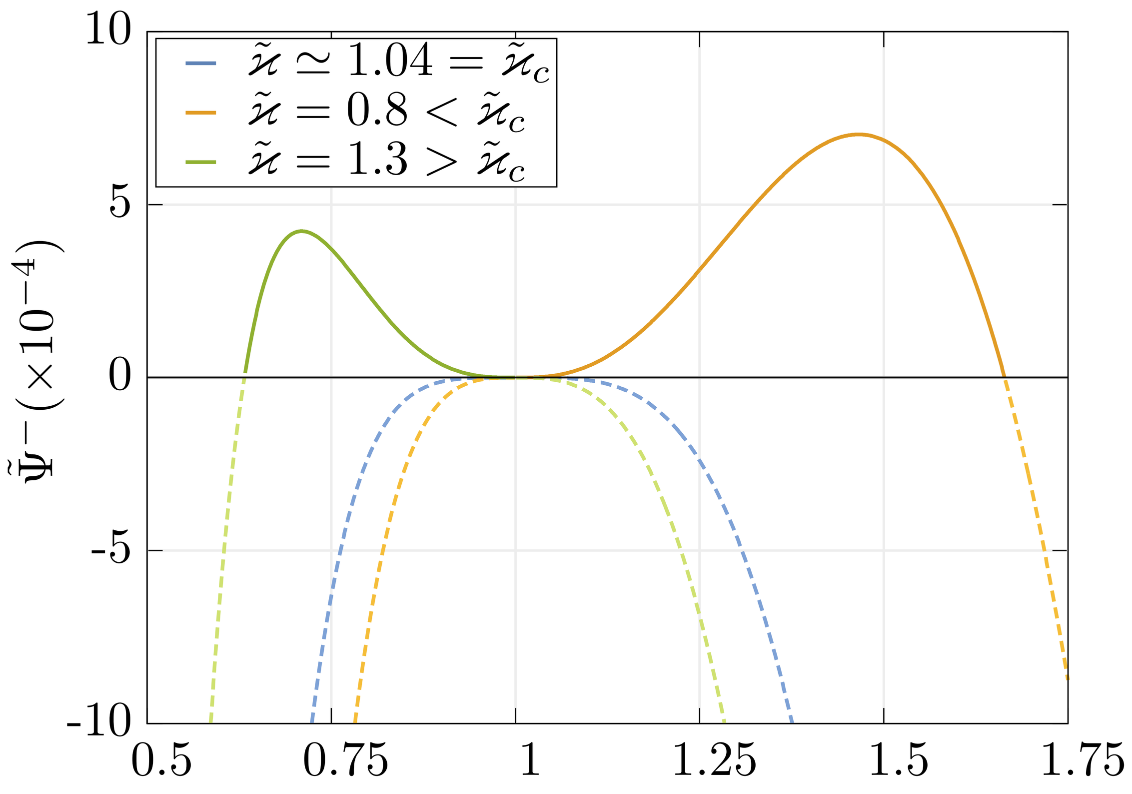

Figure 1: Plot of the function Ψ as a function of the specific volume along the Hugoniot curve for the thermal waves. Note that for $\varkappa < \varkappa_c$, $\exists$ region where $v > 1$ and $\psi \geq 0 \Rightarrow$ existence of expansion shocks.

6. Some unorthodox numerical results

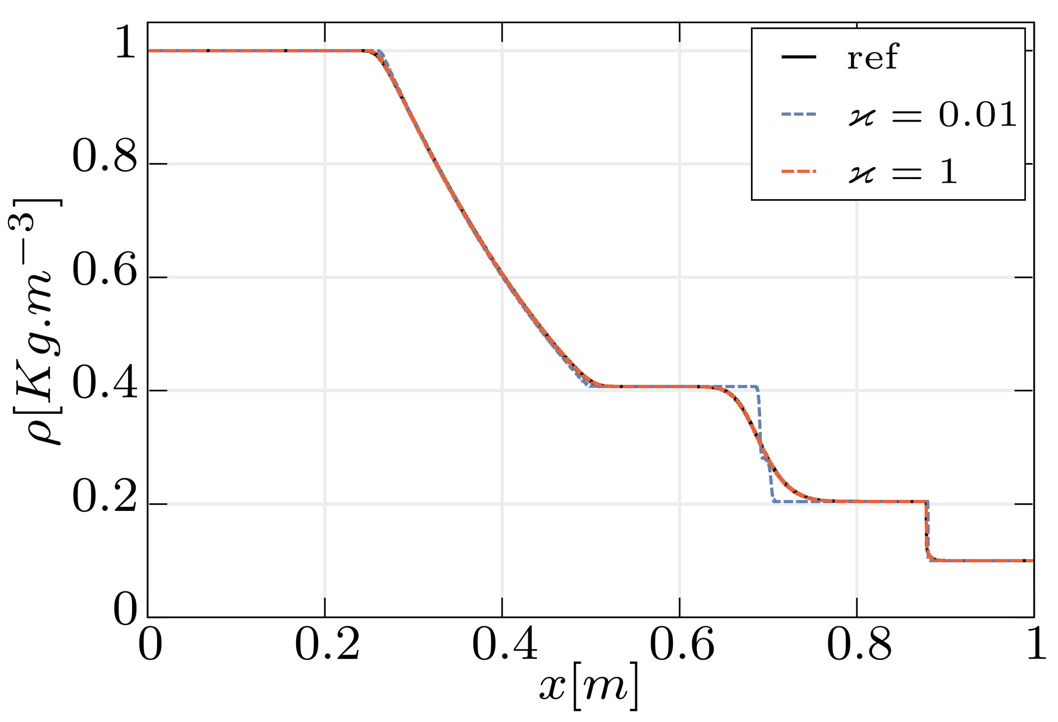

Figure 2: Comparison with Euler-Fourier system for two values of $\varkappa$ for a choc-tube problem.

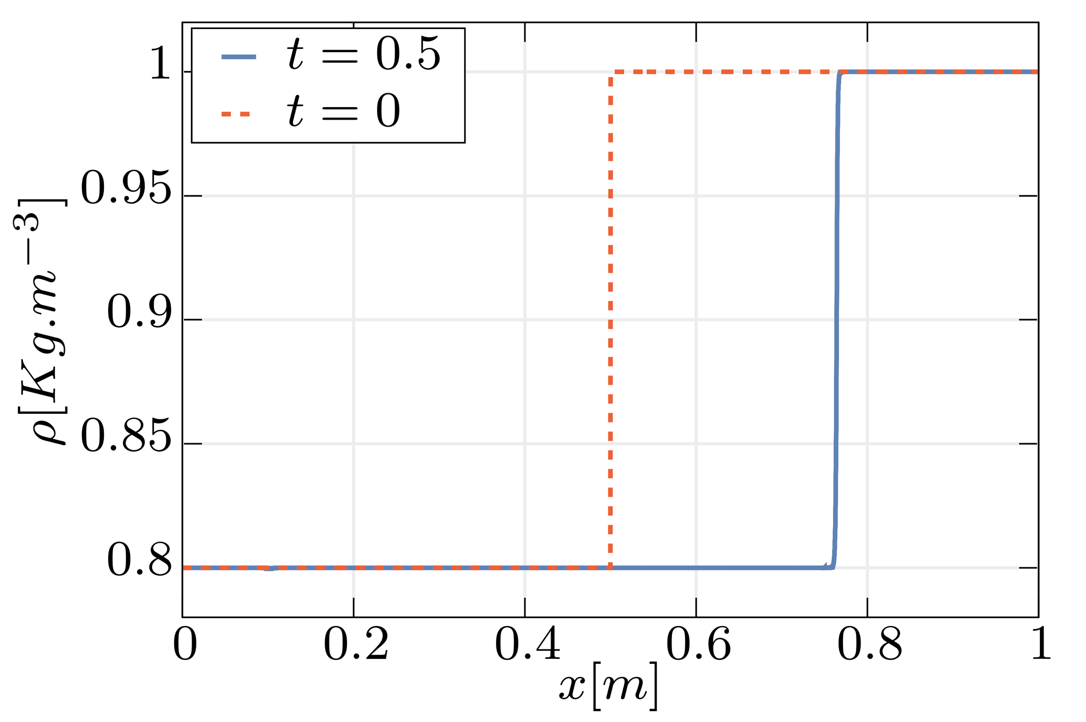

Figure 3: Expansion shock solution. Shock is advancing towards regions of higher densities.

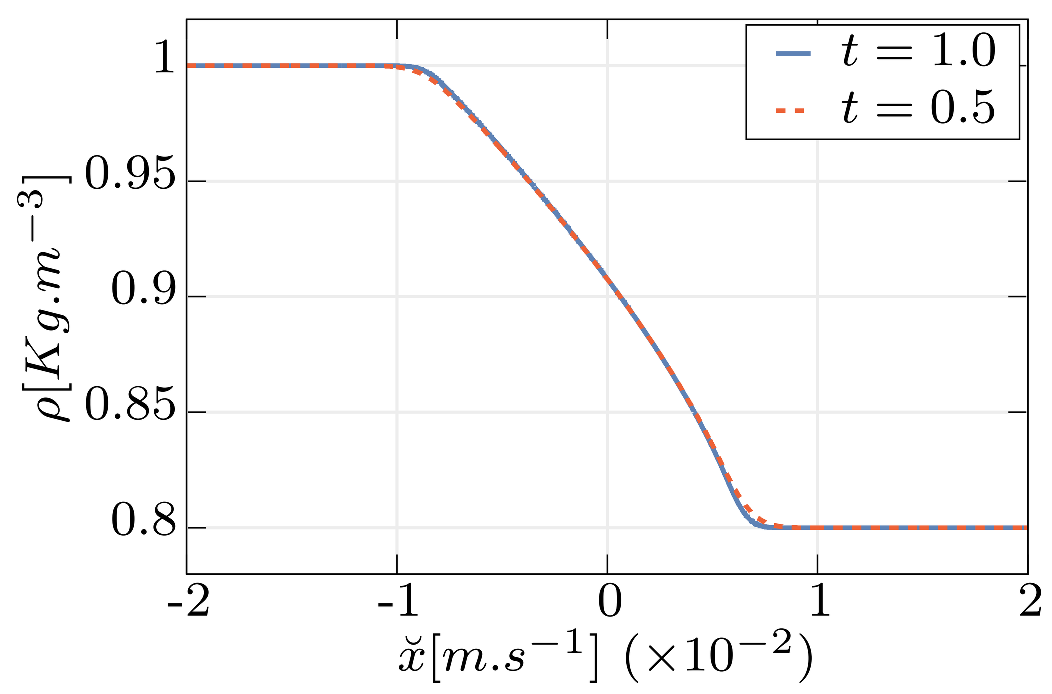

Figure 4: Compressive rarefaction solution (Velocity on the left is higher than on the right.)

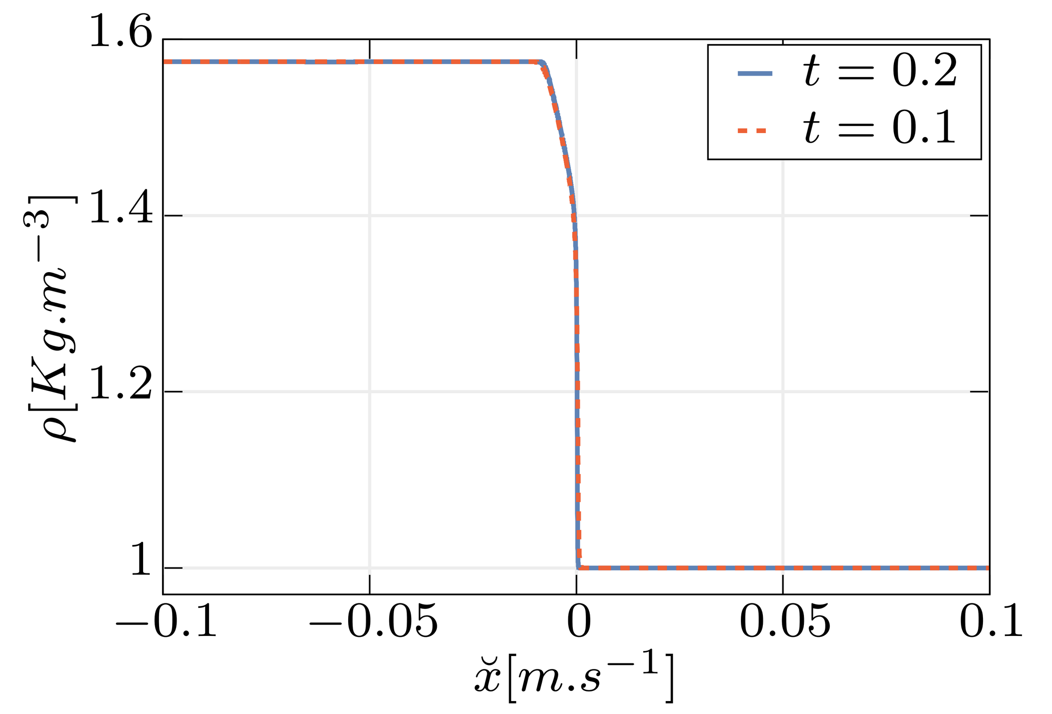

Figure 5: Shock-splitting solution. The velocity of the shock coincides with the characteristic speed behind of the shock.

7. Reference

For further details (derivation of the model, variational calculus, symmetrization, proofs, analysis , etc ...) please refer to the related preprint.

[1] Firas Dhaouadi and Sergey Gavrilyuk. An eulerian hyperbolic model for heat transfer derived via hamilton's principle: analytical and numerical study. *arXiv preprint arXiv:2305.12229*, 2023.