A structure-preserving scheme for a hyperbolic approximation to the Navier-Stokes-Korteweg equations

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Joint work with
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Main objective

Consider the Navier-Stokes-Korteweg system of equations :

$$\rho_t + \mathrm{div}(\rho \mathbf{u}) = 0$$
$$(\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = \mu \,\mathrm{div}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3}\mathrm{div}(\mathbf{u})\mathbf{I}\right) + \rho \nabla \left(K(\rho)\Delta \rho + \frac{1}{2}K'(\rho)|\nabla \rho|^2\right)$$

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- ✓ A diffuse interface option for viscous two-phase flows.

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Suggested solution

A first-order hyperbolic approximation to the NSK system!

A non-exhaustive subset of connected works and topics

1. A family of Parabolic relaxation models of NSK equations.
   - ⇒ Corli, Rohde, Schleper 2014 (DG for NSK)
   - ⇒ Hitz,Keim,Munz,Rohde 2020 (Barotropic case)
   - ⇒ Keim,Munz,Rohde 2023 [non-Isothermal NSK]

   and many other works...
2. Hyperbolic approximation of Euler-Korteweg equations.
   - ⇒ Dhaouadi, Favrie, Gavrilyuk 2019. (Schrödinger equation)
   - ⇒ Dhaouadi, Gavrilyuk, Vila 2022. (Thin films).
   - ⇒ Bourgeois, Lombard, Favrie 2020 (Solids with nonconvex EOS)
   - ⇒ Bresch *et al.*,2020 (2nd Order Hyperbolic)
3. Hyperbolic reformulation of Navier-Stokes equations.
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Idea

Combine our augmented Lagrangian model with the general *GPR* model of continuum mechanics.

Outline

Dissipationless Euler-Korteweg-Van Der Waals equations

The equations write :

$$\begin{cases} \rho_t + \mathrm{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = \rho \nabla \left(K(\rho) \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2 \right) \end{cases}$$

where $\rho = \rho(\mathbf{x}, t)$, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and $(\mathbf{x}, t) \in \mathbb{R}^d \times [0, T]$

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- $K(\rho) = \gamma$: **Compressible flow with surface tension**

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- $K(\rho) = \frac{1}{4\rho}$: **Quantum hydrodynamics**

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Lagrangian for the Euler-Korteweg-VdW system

(EK) system can be derived from the Lagrangian :

$$\mathcal{L} = \int_{\Omega_t} \left(\frac{\rho \left| \mathbf{u} \right|^2}{2} - W(\rho) - \gamma \frac{\left| \nabla \rho \right|^2}{2} \right) \, d\Omega$$

$\downarrow$ Variational principle
+
Differential constraint : $\rho_t + \mathrm{div}(\rho \mathbf{u}) = 0$

$$(\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla(P(\rho)) = \gamma \rho \nabla(\Delta \rho)$$

with $P(\rho) = \rho W'(\rho) - W(\rho)$

Augmented Lagrangian approach

$$\mathcal{L}(\mathbf{u}, \rho, \nabla\rho) = \int_{\Omega_t} \left(\frac{1}{2}\rho \left|\mathbf{u}\right|^2 - W(\rho) - \gamma \frac{\left|\nabla\rho\right|^2}{2} \right) d\Omega$$

$$\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0$$

'Augmented' Lagrangian approach [Favrie-Gavrilyuk 2017]

$$\tilde{\mathcal{L}}(\mathbf{u}, \rho, \eta, \nabla\eta) \qquad (\eta \longrightarrow \rho)$$

$$\tilde{\mathcal{L}} = \int_{\Omega_t} \left(\rho \frac{\left|\mathbf{u}\right|^2}{2} - W(\rho) - \gamma \frac{\left|\nabla\eta\right|^2}{2} - \frac{1}{2\alpha\rho} (\rho - \eta)^2 \right) d\Omega$$

$\frac{1}{2\alpha\rho} (\rho - \eta)^2$: Classical Penalty term

Preliminary system

By applying Hamilton's principle for the Eulerian variations $\delta \mathbf{x}$ and $\delta \eta$ one obtains the system of governing equations

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where:

$$\mathbf{K}_\alpha = \left(\frac{\gamma}{2}|\nabla \eta|^2 - \frac{\eta}{\alpha}\left(1 - \frac{\eta}{\rho}\right)\right)\mathbf{Id} - \gamma \nabla \eta \otimes \nabla \eta$$

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Replacing the relaxation term in the stress tensor yields

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$\mathrm{div}(\mathbf{K}_\alpha) = \gamma\eta\nabla(\Delta\eta)$, original: $\mathrm{div}(\mathbf{K}) = \gamma\rho\nabla(\Delta\rho)$

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Idea : Include $\dot{\eta}$ into the Lagrangian !

Augmented Lagrangian - Attempt 2

Augmented Lagrangian approach

$$\tilde{\mathcal{L}}(\mathbf{u}, \rho, \eta, \nabla\eta, \dot{\eta}) \qquad \alpha, \beta \ll 1$$

$$\tilde{\mathcal{L}} = \int_{\Omega_t} \left(\rho \frac{|\mathbf{u}|^2}{2} - W(\rho) - \frac{\gamma}{2}|\nabla\eta|^2 - \frac{1}{2\alpha\rho}(\rho - \eta)^2 + \frac{\beta\rho}{2}\dot{\eta}^2 \right) d\Omega$$

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$\downarrow$ Variational principle : $a = \int_{t_0}^{t_1} \tilde{\mathcal{L}} \, dt$

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$\Rightarrow$ There are still high-order derivatives!

Order reductions

1. We take $w = \dot{\eta}$ as independent variable. Thus:

$$w = \eta_t + \mathbf{u} \cdot \nabla \eta \quad \Longrightarrow \quad \boxed{(\rho\eta)_t + \mathrm{div}(\rho\eta\mathbf{u}) = \rho w}$$

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$$\nabla w = \nabla(\eta_t + \mathbf{u} \cdot \nabla \eta)$$

$$\Longrightarrow \quad \boxed{\mathbf{p}_t + \nabla(\mathbf{p} \cdot \mathbf{u} - w) = 0}$$

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- Important: Initial data must be such that: $\mathbf{p}(\mathbf{x}, 0) = \nabla\eta(\mathbf{x}, 0), \quad w(\mathbf{x}, 0) = \dot{\eta}(\mathbf{x}, 0)$

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- IMPORTANT: $\mathbf{p} = \nabla \eta \;\Rightarrow\; \nabla \times \mathbf{p} \equiv 0$

Final form of the approximate Euler-Korteweg system

$$\begin{cases} \rho_t & + \operatorname{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t & + \operatorname{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + P(\rho)\mathbf{Id} - \mathbf{K}_\alpha\right) = 0 \\ (\rho w)_t & + \operatorname{div}\left(\rho w \mathbf{u} - \gamma \mathbf{p}\right) = \frac{1}{\alpha\beta}\left(1 - \frac{\eta}{\rho}\right) \\ \mathbf{p}_t & + \nabla\left(\mathbf{p} \cdot \mathbf{u} - w\right) \;= 0 \\ (\rho\eta)_t & + \operatorname{div}(\rho\eta\mathbf{u}) = \rho w \end{cases}$$

$$\mathbf{K}_\alpha = \left(\frac{\gamma}{2}|\mathbf{p}|^2 - \frac{\eta}{\alpha}\left(1 - \frac{\eta}{\rho}\right)\right)\mathbf{Id} - \gamma \mathbf{p} \otimes \mathbf{p}$$

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\mathbf{p}_t & + \nabla(\mathbf{p}\cdot\mathbf{u} - w) \ = 0 \\
(\rho\eta)_t & + \operatorname{div}(\rho\eta\mathbf{u}) = \rho w
\end{cases}
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But recall that $\mathbf{p} = \nabla\eta \ \Rightarrow \ \nabla \times \mathbf{p} = 0$ ...

Final form of the approximate Euler-Korteweg system

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \operatorname{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + P(\rho)\mathbf{Id} - \mathbf{K}_\alpha\right) = 0 \\ (\rho w)_t + \operatorname{div}\left(\rho w \mathbf{u} - \gamma \mathbf{p}\right) = \frac{1}{\alpha\beta}\left(1 - \frac{\eta}{\rho}\right) \\ \mathbf{p}_t + \nabla(\mathbf{p}\cdot\mathbf{u} - w) + (\nabla \times \mathbf{p}) \times \mathbf{u} = 0 \\ (\rho\eta)_t + \operatorname{div}(\rho\eta\mathbf{u}) = \rho w \end{cases}$$

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$\Rightarrow$ Now the system is Gallilean invariant...

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But recall that $\mathbf{p} = \nabla\eta \;\Rightarrow\; \nabla \times \mathbf{p} = 0$ ...
$\Rightarrow$ Now the system is Gallilean invariant... But is it hyperbolic ?

Hyperbolicity in 1-D

$\mathbf{A}$ admits 5 eigenvalues that can be expressed as follows :
Reminder ($P(\rho)$: hydrostatic pressure, $p = \eta_x$)

$$\xi = \begin{pmatrix} u \\ u + \sqrt{\psi_1 + \psi_2} \\ u + \sqrt{\psi_1 - \psi_2} \\ u - \sqrt{\psi_1 + \psi_2} \\ u - \sqrt{\psi_1 - \psi_2} \end{pmatrix} \text{ with } \begin{cases} \psi_1 = \frac{1}{2}(a^2 + a_\gamma^2 + a_\alpha^2 + a_\beta^2) \\ \psi_2 = \frac{1}{2}\sqrt{(a^2 + a_\gamma^2 + a_\alpha^2 - a_\beta^2)^2 + 4a_\beta^2 a_\gamma^2} \\ a = \sqrt{P'(\rho)}, \quad a_\gamma = \sqrt{\dfrac{\gamma}{\rho}p^2} \\ a_\alpha = \dfrac{\eta}{\rho\sqrt{\alpha}}, \quad a_\beta = \sqrt{\dfrac{\gamma}{\beta\rho}} \end{cases}$$

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a^2: adiabatic sound speed.
a_γ: wave speed due to capillarity .
a_α and a_β: First and second relaxation speeds.

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a^2: adiabatic sound speed. (negative in non-convex regions!!)
a_γ: wave speed due to capillarity .
a_α and a_β: First and second relaxation speeds.

Van der Waals equation of state

In the context of two-phase flows, the equation of state is non-convex

$$p = \frac{\rho R T}{1 - b\rho} - a\rho^2, \qquad a > 0,\ b > 0$$

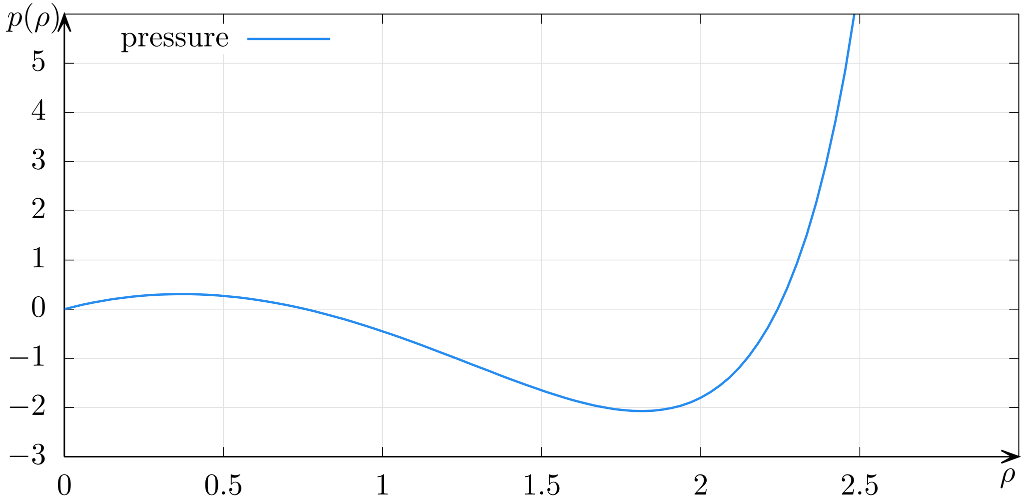


Figure 1: Van der Waals pressure for $T = 0.85, a = 3, b = 1/3, R = 8/3$

Navier-Stokes-Korteweg equations

In general, the equations write

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = \underline{\underline{S}} + \underline{\underline{K}} \end{cases}$$

where $\rho = \rho(\mathbf{x}, t)$, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and $(\mathbf{x}, t) \in \mathbb{R}^d \times [0, T]$
The (dispersive) Korteweg stress tensor is given by:

$$\underline{\underline{K}} = \rho \nabla \left(\gamma \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2 \right)$$

and the (viscous) Navier-Stokes stresses are given by

$$\underline{\underline{S}} = \mu \operatorname{div} \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3} \operatorname{div}(\mathbf{u}) \mathbf{I} \right)$$

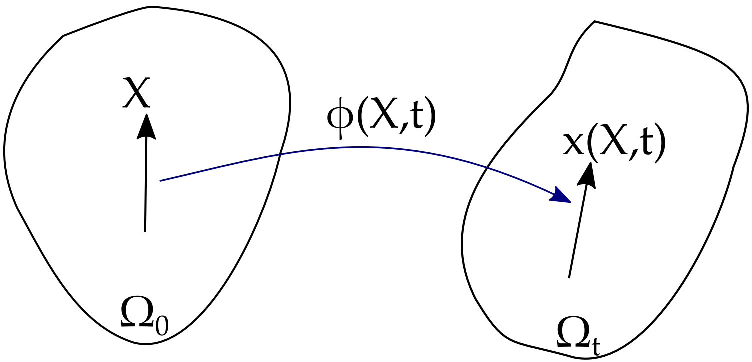
Godunov-Peshkov-Romenski Model of continuum mechanics

Deformation gradient:

$$\mathbf{F} = \left[\frac{\partial x_i}{\partial X_j}\right]$$

Inverse Deformation gradient:

$$\mathbf{A} = \mathbf{F}^{-1} = \left[\frac{\partial X_i}{\partial x_j}\right]$$



$$\partial_t(\mathbf{A}) + \nabla(\mathbf{A}\mathbf{u}) + \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}}\right)^T\right)\cdot \mathbf{u} = 0 \quad \text{(Solids)}$$

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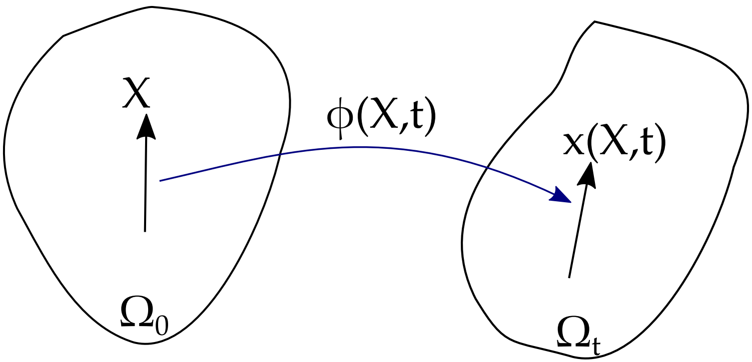

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$$\partial_t(\mathbf{A}) + \nabla(\mathbf{A}\mathbf{u}) + \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}}\right)^T\right) \cdot \mathbf{u} = 0 \quad \text{(Solids)}$$

$$\partial_t(\mathbf{A}) + \nabla(\mathbf{A}\mathbf{u}) + \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}}\right)^T\right) \cdot \mathbf{u} = \frac{1}{\tau}\,\mathbf{S}(\mathbf{A}) \quad \text{(Fluids)}$$

Hyperbolic NSK = Hyperbolic EK + Hyperbolic NS

(Black: Euler part, Red: Dispersive part, Blue: Viscous part.)

$$\partial_t(\rho) + \mathrm{div}(\rho\mathbf{u}) = 0$$

$$\partial_t(\rho\mathbf{u}) + \mathrm{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + P(\rho)\mathbf{Id} - K_\alpha - \sigma\right) = 0$$

$$\partial_t(\rho\eta) + \mathrm{div}(\rho\eta\mathbf{u}) = \rho w$$

$$\partial_t(\rho w) + \mathrm{div}\left(\rho w\mathbf{u} - \gamma\mathbf{p}/\beta\right) = (\alpha\beta)^{-1}\left(1 - \eta/\rho\right)$$

$$\partial_t(\mathbf{p}) + \nabla\left(\mathbf{p}\cdot\mathbf{u} - w\right) + (\nabla\times\mathbf{p})\times\mathbf{u} = 0,$$

$$\partial_t(\mathbf{A}) + \nabla(\mathbf{A}\mathbf{u}) + \left(\frac{\partial\mathbf{A}}{\partial\mathbf{x}} - \left(\frac{\partial\mathbf{A}}{\partial\mathbf{x}}\right)^T\right)\cdot\mathbf{u} = -\frac{3}{\tau}\det(\mathbf{A})^{5/3}\mathbf{A}\,\mathrm{dev}(\mathbf{G})$$

$$\text{where}\begin{cases}\sigma = -\rho c_s^2\mathbf{G}\,\mathrm{dev}(\mathbf{G}) \text{ where } \mathbf{G} = \mathbf{A}^T\mathbf{A}\\ \mathbf{K}_\alpha = -\gamma\mathbf{p}\otimes\mathbf{p} + \left(\frac{\gamma}{2}|\mathbf{p}|^2 - \frac{\eta}{\alpha}\left(1 - \frac{\eta}{\rho}\right)\right)\mathbf{Id}\end{cases}$$

Eigenvalues - Hyperbolicity

$\Rightarrow$ 18 Real Eigenvalues (Linearized around $A = \mathbf{I}, \mathbf{p} = (p1, 0, 0)^T$)

Transport: $\lambda_{1-10} = u_1,$

shear waves: $\begin{cases} \lambda_{11-12} = u_1 + c_s, \\ \lambda_{13-14} = u_1 - c_s, \end{cases}$

Mixed waves:

$$\begin{cases} \lambda_{15} = u_1 - \sqrt{Z_1 + Z_2} \\ \lambda_{16} = u_1 - \sqrt{Z_1 - Z_2} \\ \lambda_{17} = u_1 + \sqrt{Z_1 + Z_2} \\ \lambda_{18} = u_1 + \sqrt{Z_1 - Z_2} \end{cases}, \begin{cases} Z_1 = \frac{1}{2}(a_0^2 + a_s^2 + a_\gamma^2 + a_\alpha^2 + a_\beta^2), \\ Z_2 = \sqrt{Z_1^2 - a_\beta^2(a_0^2 + a_\alpha^2 + a_s^2)}, \\ a_0 = \sqrt{\rho W''(\rho)}, \quad a_s = \sqrt{\frac{4}{3}c_s^2} \\ a_\alpha = \frac{\eta}{\rho\sqrt{\alpha}}, \quad a_\beta = \sqrt{\frac{\gamma}{\beta\rho}}, \quad a_\gamma = \sqrt{\frac{\gamma}{\rho}p_1^2} \end{cases}$$

Eigenvalues - Hyperbolicity

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Transport: $\lambda_{1-10} = u_1,$ $\Rightarrow$Missing 2 eigenvectors !

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Strong Hyperbolicity?

$$\begin{aligned}
\partial_t(\rho) \quad & + \operatorname{div}(\rho \mathbf{u}) = 0 \\
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\partial_t(\mathbf{p}) \quad & + \nabla\left(\mathbf{p} \cdot \mathbf{u} - w\right) + (\nabla \times \mathbf{p}) \times \mathbf{u} = 0, \\
\partial_t(\mathbf{A}) \quad & + \nabla(\mathbf{A}\mathbf{u}) + \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}}\right)^T\right) \cdot \mathbf{u} = -\frac{3}{\tau} \det(\mathbf{A})^{5/3} \mathbf{A} \operatorname{dev}(\mathbf{G})
\end{aligned}$$

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$\Rightarrow$ Recall that $\mathbf{p} = \nabla\eta \;\Rightarrow\; \nabla\times\mathbf{p} = 0$ ...

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$\Rightarrow$ Recall that $\mathbf{p} = \nabla\eta \;\Rightarrow\; \nabla\times\mathbf{p} = 0$ ...

$\Rightarrow$ We can add a "zero" to restore strong hyperbolicity.

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$\Rightarrow$ We can add a "zero" to restore strong hyperbolicity.

$\Rightarrow$ This procedure is also used to symmetrize such systems (Godunov-Powell symmetrizing terms).

System to be solved numerically

A set of classical conservation laws:

$$
\begin{aligned}
&\partial_t(\rho) && + \operatorname{div}(\rho \mathbf{u}) = 0 \\
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\end{aligned}
$$

A set of potentially curl constrained vectors:

$$
\begin{aligned}
&\partial_t(\mathbf{p}) && + \nabla\left(\mathbf{p} \cdot \mathbf{u} - w\right) = 0, \\
&\partial_t(\mathbf{A_1}) && + \nabla(\mathbf{A_1} \cdot \mathbf{u}) + (\nabla \times \mathbf{A}_1) \times \mathbf{u} = -\frac{1}{\tau}\mathbf{S}_1 \\
&\partial_t(\mathbf{A_2}) && + \nabla(\mathbf{A_2} \cdot \mathbf{u}) + (\nabla \times \mathbf{A}_2) \times \mathbf{u} = -\frac{1}{\tau}\mathbf{S}_2 \\
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\end{aligned}
$$

System to be solved numerically

A set of classical conservation laws: MUSCL-Hancock FV scheme

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&\partial_t(\rho) && + \operatorname{div}(\rho\mathbf{u}) = 0\\
&\partial_t(\rho\mathbf{u}) && + \operatorname{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + P(\rho)\mathbf{Id} - K_\alpha - \sigma\right) = 0\\
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\end{aligned}$$

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\partial_t(\rho w) &\quad + \mathrm{div}\left(\rho w\mathbf{u} - \gamma\mathbf{p}/\beta\right) = (\alpha\beta)^{-1}\left(1 - \eta/\rho\right)
\end{aligned}$$

A set of potentially curl constrained vectors: VIP Treatment

$$\begin{aligned}
\partial_t(\mathbf{p}) &\quad + \nabla\left(\mathbf{p}\cdot\mathbf{u} - w\right) = 0, \\
\partial_t(\mathbf{A_1}) &\quad + \nabla(\mathbf{A_1}\cdot\mathbf{u}) + (\nabla\times\mathbf{A}_1)\times\mathbf{u} = -\frac{1}{\tau}\mathbf{S}_1 \\
\partial_t(\mathbf{A_2}) &\quad + \nabla(\mathbf{A_2}\cdot\mathbf{u}) + (\nabla\times\mathbf{A}_2)\times\mathbf{u} = -\frac{1}{\tau}\mathbf{S}_2 \\
\partial_t(\mathbf{A_3}) &\quad + \nabla(\mathbf{A_3}\cdot\mathbf{u}) + (\nabla\times\mathbf{A}_3)\times\mathbf{u} = -\frac{1}{\tau}\mathbf{S}_3
\end{aligned}$$

Exactly curl-free scheme: Staggered Grid

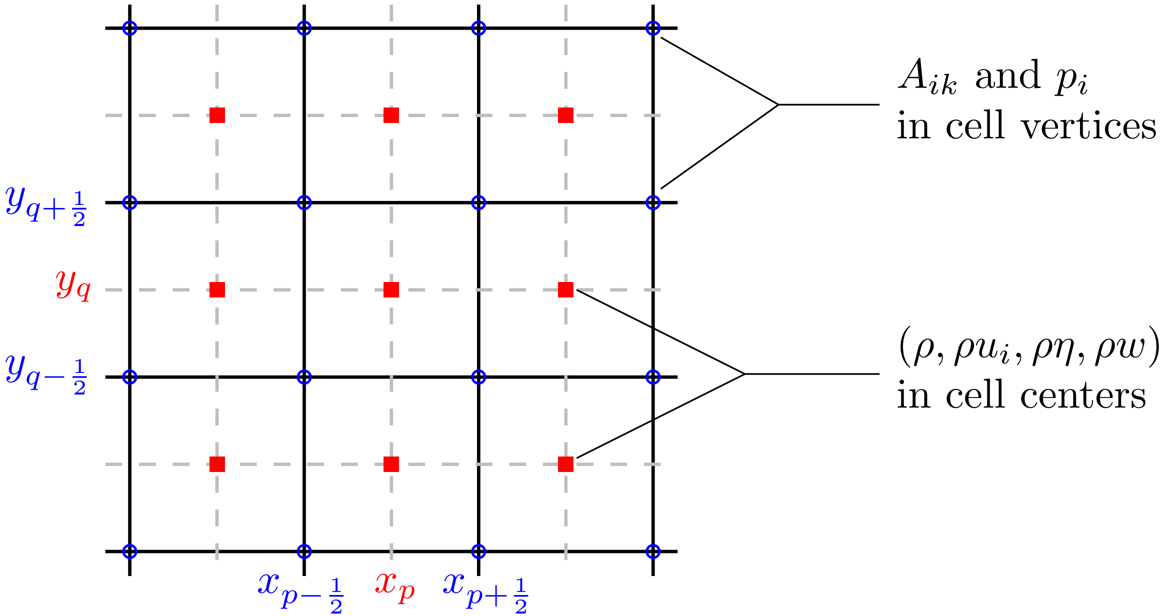
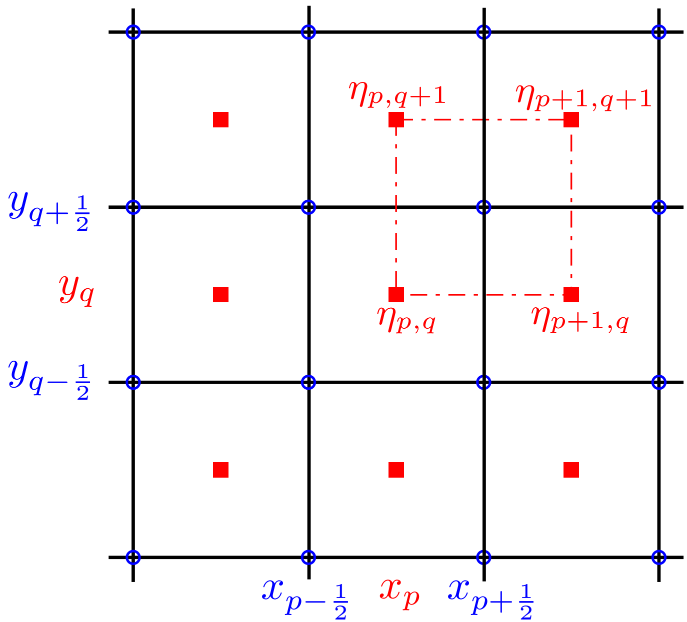


Figure 2: Schematic of the computational domain featuring the grid points and the staggered dual grid points. Red squares are barycenters and blue circles are the vertexes of the computational cells.

Exactly curl-free scheme: Compatible gradient stencil



$$\begin{cases} (\partial_x^h \phi)^{p+\frac{1}{2},q+\frac{1}{2}} = \dfrac{1}{2}\dfrac{\phi^{p+1,q} - \phi^{p,q}}{\Delta x} + \dfrac{1}{2}\dfrac{\phi^{p+1,q+1} - \phi^{p,q+1}}{\Delta x}, \\ (\partial_y^h \phi)^{p+\frac{1}{2},q+\frac{1}{2}} = \dfrac{1}{2}\dfrac{\phi^{p,q+1} - \phi^{p,q}}{\Delta y} + \dfrac{1}{2}\dfrac{\phi^{p+1,q+1} - \phi^{p+1,q}}{\Delta y}. \end{cases}$$

Figure 3: Stencil of the gradient field computed in every corner

Exactly curl-free scheme: Compatible curl stencil

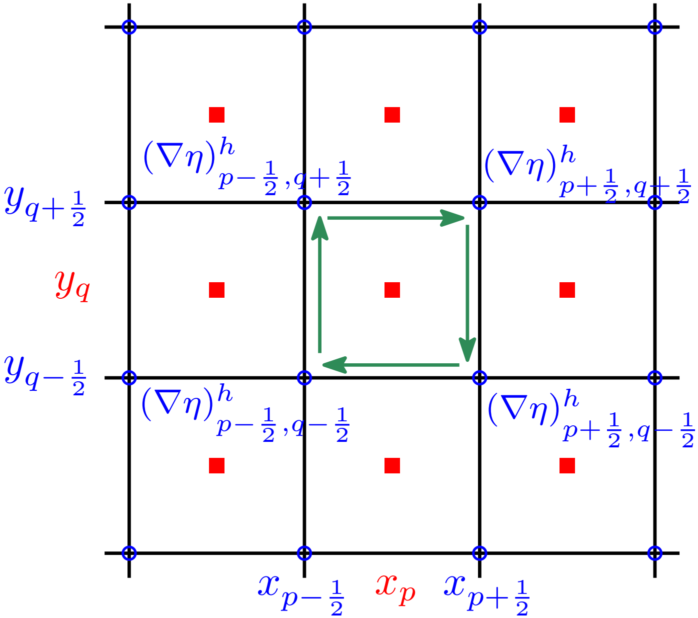


Figure 4: Stencil of the curl operator computed in every cell-center

Compatible discrete curl-operator

Based on this corner gradient, one can now define a compatible discrete curl operator such that $(\nabla^h \times \nabla^h \phi)^{p,q} \cdot \mathbf{e_z}$ is given by

$$\frac{(\partial_y^h \phi)^{p+\frac{1}{2},q+\frac{1}{2}} - (\partial_y^h \phi)^{p+\frac{1}{2},q-\frac{1}{2}}}{2\Delta y} + \frac{(\partial_y^h \phi)^{p-\frac{1}{2},q+\frac{1}{2}} - (\partial_y^h \phi)^{p-\frac{1}{2},q-\frac{1}{2}}}{2\Delta y}$$
$$- \frac{(\partial_x^h \phi)^{p+\frac{1}{2},q+\frac{1}{2}} - (\partial_x^h \phi)^{p-\frac{1}{2},q+\frac{1}{2}}}{2\Delta x} - \frac{(\partial_x^h \phi)^{p+\frac{1}{2},q-\frac{1}{2}} - (\partial_x^h \phi)^{p-\frac{1}{2},q-\frac{1}{2}}}{2\Delta x}.$$

It is straightforward to prove that

$$\nabla^h \times \nabla^h \phi \equiv 0$$

Update formulas ($h = \min(\Delta x, \Delta y)$)

- For the conserved variables $\rho, \mathbf{u}, \rho\eta, \rho w$:
  $\Rightarrow$ Classical MUSCL-Hancock scheme.

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$$p_k^{p+\frac{1}{2},q+\frac{1}{2},n+1} = p_k^{p+\frac{1}{2},q+\frac{1}{2},n} - \Delta t\, \nabla_k^h \left(p_j u_j - w\right)^{p+\frac{1}{2},q+\frac{1}{2},n}$$

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- Lastly, for $\mathbf{A}$

$$\begin{aligned} A_{ik}^{p+\frac{1}{2},q+\frac{1}{2},n+1} = A_{ik}^{p+\frac{1}{2},q+\frac{1}{2},n} &- \Delta t(\nabla_k^h(A_{ij}u_j) - h\ c^* \nabla_j^h A_{ij})^{p+\frac{1}{2},q+\frac{1}{2}} \\ &- \Delta t\ h\ c^* \varepsilon_{kj3} \nabla_j^{p+\frac{1}{2},q+\frac{1}{2},n} \left(\varepsilon_{3lm} \nabla_l^h A_{im} \right) \\ &- \frac{\Delta t}{4} \sum_{r=0}^{1} \sum_{s=0}^{1} u_m^{p+r,q+s,n} \left((\nabla_m^h A_{ik})^{p+\frac{1}{2},q+\frac{1}{2}} - (\nabla_k^h A_{im})^{p+\frac{1}{2},q+\frac{1}{2}} \right) \\ &- \Delta t \frac{1}{3\tau} \det(\mathbf{A}^{p+\frac{1}{2},q+\frac{1}{2},n+1})^{5/3} A_{im}^{p+\frac{1}{2},q+\frac{1}{2},n+1} \mathring{G}_{mk}^{p+\frac{1}{2},q+\frac{1}{2},n+1}. \end{aligned}$$

Near equilibrium bubble: density field

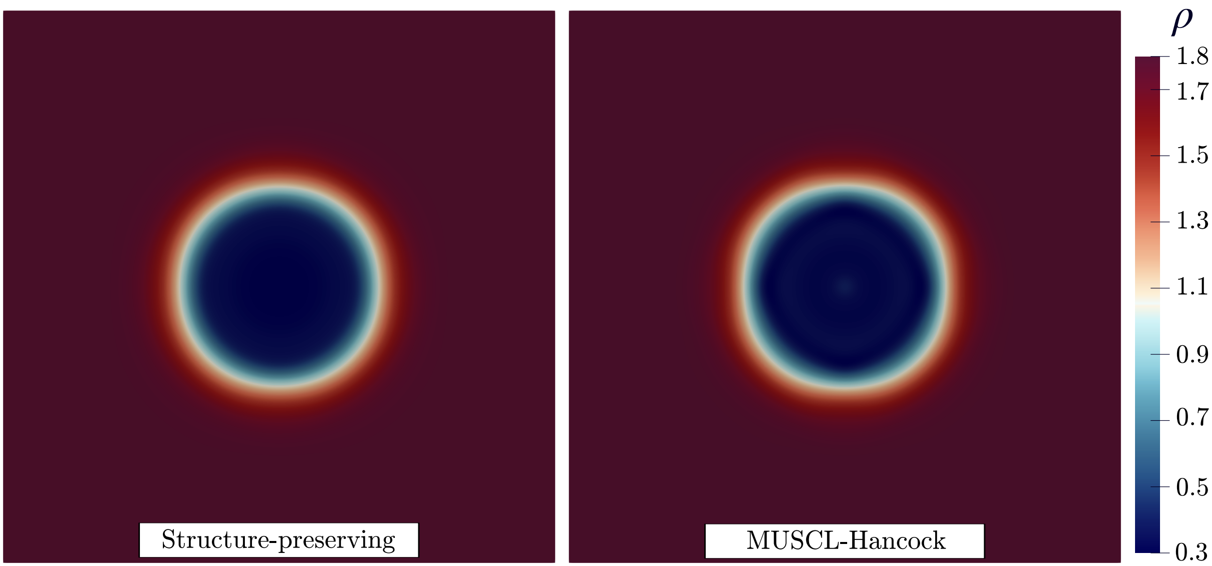

Figure 5: Results are shown for $t = 2$ on a 512×512 grid. With $\gamma = 2.10^{-4}$, $\alpha = 10^{-2}$, $\beta = 10^{-5}$, $\mu = 10^{-2}, c_s = 10$. The computational domain is $\Omega_c = [-0.25, 0.25] \times [-0.25, 0.25]$.

Near equilibrium bubble: gradient field

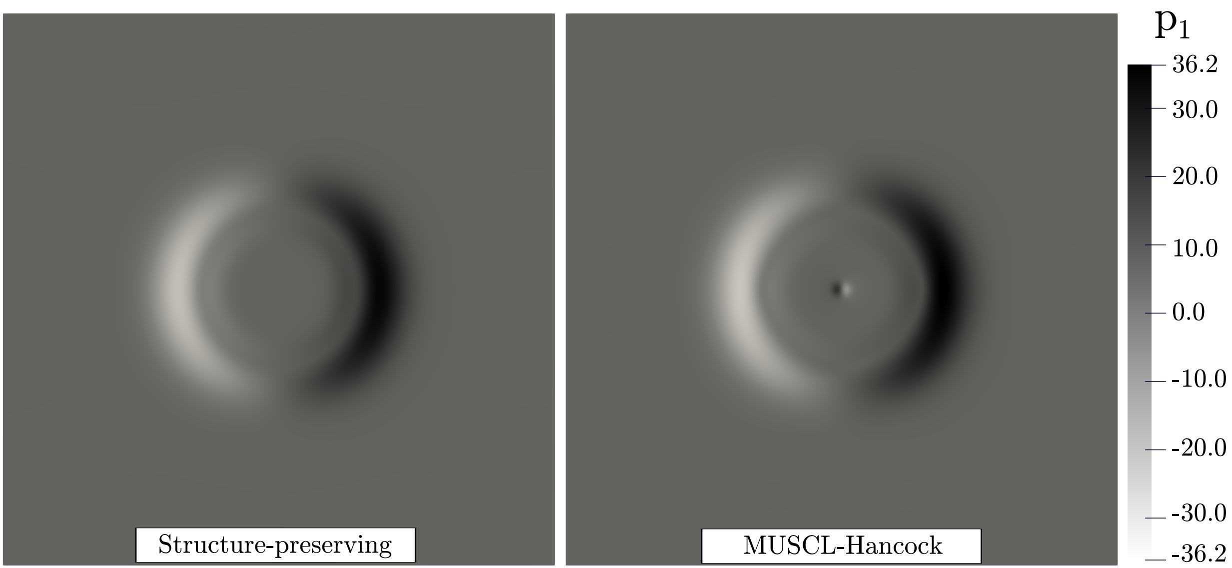

Figure 6: Results are shown for $t = 2$ on a 512×512 grid. With $\gamma = 2.10^{-4}$, $\alpha = 10^{-2}$, $\beta = 10^{-5}$, $\mu = 10^{-2}, c_s = 10$. The computational domain is $\Omega_c = [-0.25, 0.25] \times [-0.25, 0.25]$.

Near equilibrium bubble: Discrete curl error over time

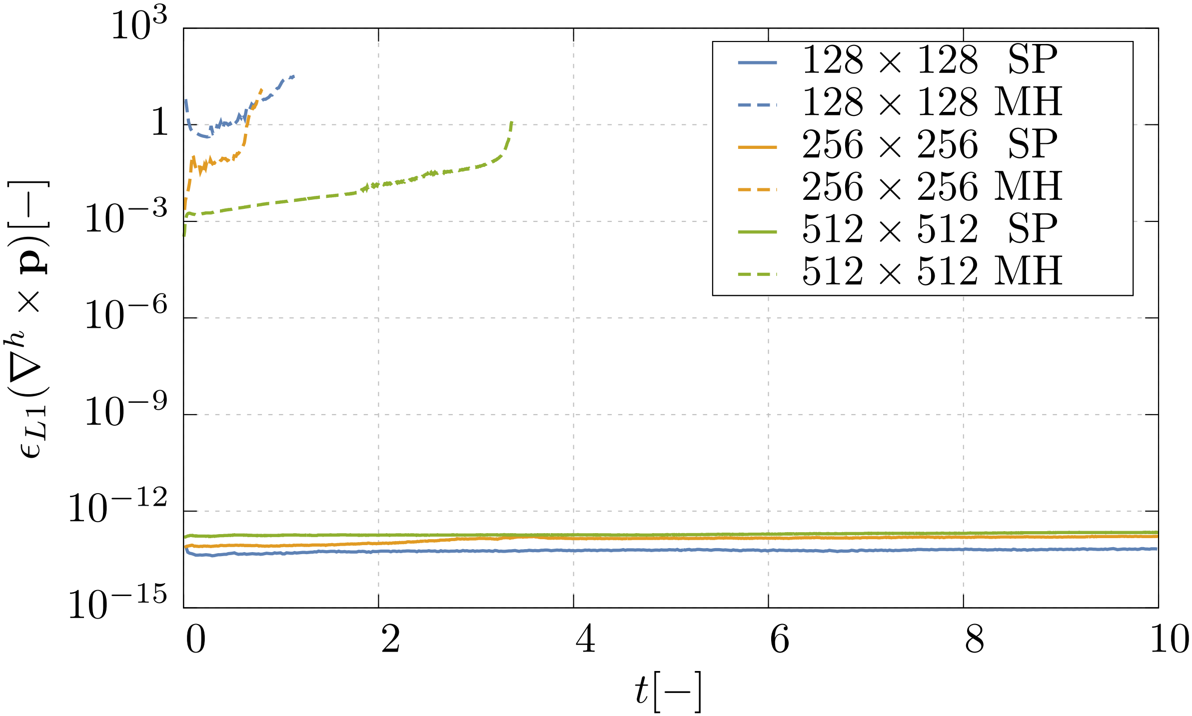

Figure 7: Time-evolution of the L_1 norm of the discrete curl errors on different mesh sizes.

2D Ostwald Ripening

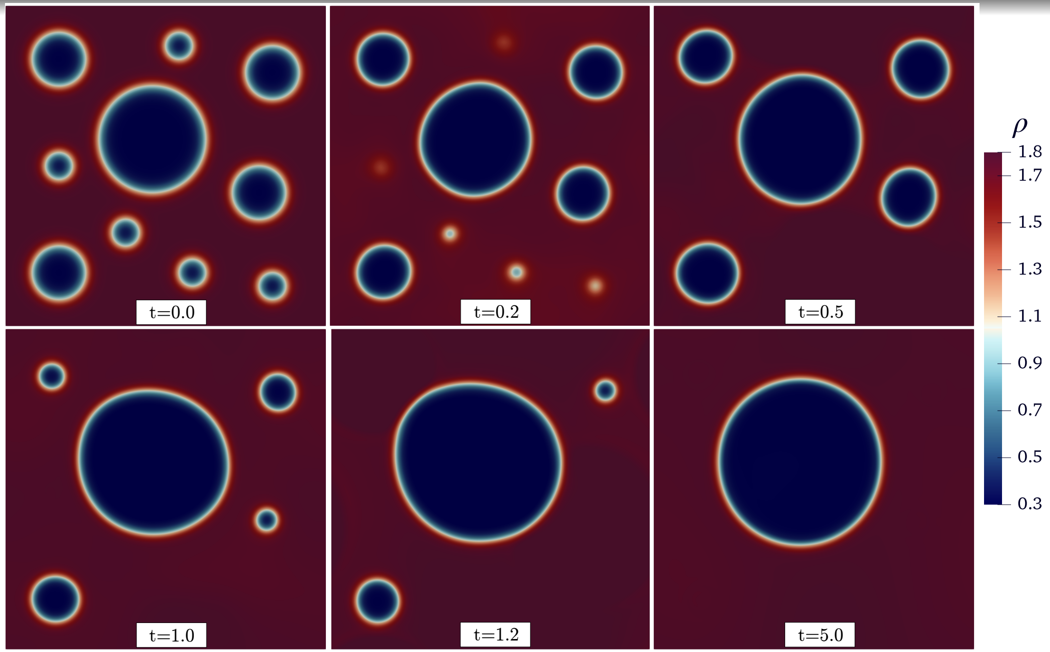

Figure 8: Values used here are $\rho_l = 1.8$, $\rho_v = 0.3$, $\gamma = 2.10^{-4}$, $\alpha = 10^{-2}$, $\beta = 10^{-5}$, $c_s = 10$ and an effective viscosity of $\mu = 10^{-2}$. The total domain is $\Omega = [-0.6, +0.6] \times [-0.6, +0.6]$ discretized over a 4096×4096 uniform grid with periodic boundary conditions.

Conclusion and Perspectives

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- We conceived a hyperbolic relaxation model to the Navier-Stokes-Korteweg equations.
- The used scheme preserves the curl errors up to machine precision.
- Some numerical results blow up in finite time if a curl-free discretization is not used

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Perspectives

- Extension to non-isothermal flows (by using the hyperbolic heat system presented by Sergey).
- Splitting of the fluxes for semi-implicit discretization
- Higher-order extension of the scheme
- Investigation of Laplace jumps... etc

Thank you for your attention !

[1] Dhaouadi, Firas, and Michael Dumbser. "A first order hyperbolic reformulation of the Navier-Stokes-Korteweg system based on the GPR model and an augmented Lagrangian approach." *Journal of Computational Physics* 470 (2022): 111544.

[2] Dhaouadi, Firas, and Michael Dumbser. "A Structure-Preserving Finite Volume Scheme for a Hyperbolic Reformulation of the Navier–Stokes–Korteweg Equations." *Mathematics* 11.4 (2023): 876.

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And references therein.

Dispersion relation

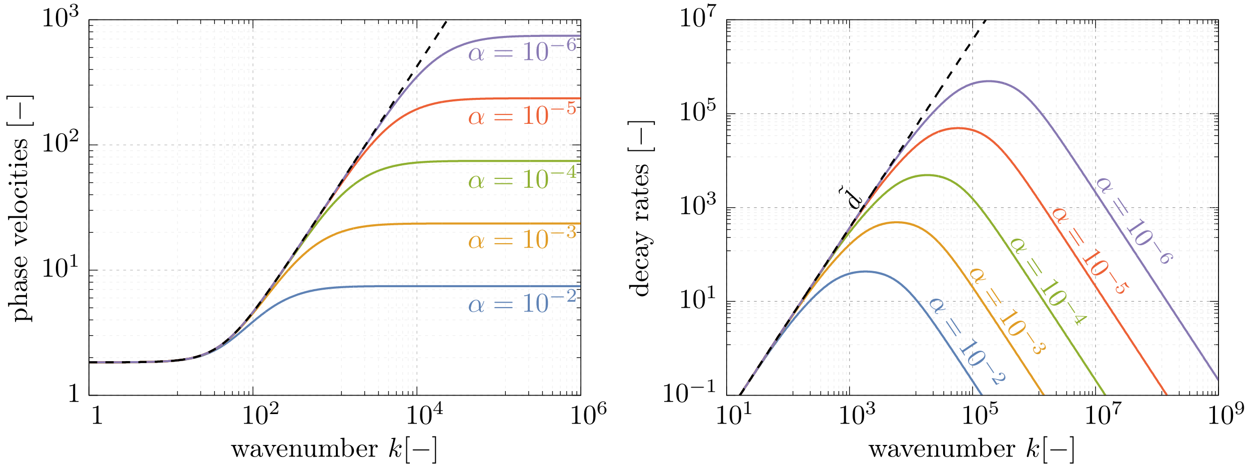

Figure 9: Plot of the phase velocity (left) and the decay rate for several values of α along their counterparts for the Navier-Stokes-Korteweg system. The model parameters are as follows $\gamma = 10^{-3}$, $\mu = 10^{-3}$ and $\rho = 1.8$

Scaling of relaxations

Representative characteristic velocities

$$\begin{cases} \lambda_{15} = u_1 - \sqrt{Z_1 + Z_2} \\ \lambda_{16} = u_1 - \sqrt{Z_1 - Z_2} \\ \lambda_{17} = u_1 + \sqrt{Z_1 + Z_2} \\ \lambda_{18} = u_1 + \sqrt{Z_1 - Z_2} \end{cases}, \quad \begin{cases} Z_1 = \frac{1}{2}(a_0^2 + a_s^2 + a_\gamma^2 + a_\alpha^2 + a_\beta^2), \\ Z_2 = \sqrt{Z_1^2 - a_\beta^2(a_0^2 + a_\alpha^2 + a_s^2)}, \\ a_0 = \sqrt{\rho W''(\rho)}, \quad a_s = \sqrt{\frac{4}{3}c_s^2} \\ a_\alpha = \frac{\eta}{\rho\sqrt{\alpha}}, \quad a_\beta = \sqrt{\frac{\gamma}{\beta\rho}}, \quad a_\gamma = \sqrt{\frac{\gamma}{\rho}p_1^2} \end{cases}$$

The different relaxation contributions scale as

$$a_\alpha^2 \sim \frac{1}{\alpha}, \quad a_\beta^2 \sim \frac{\gamma}{\beta\rho}, \quad a_s^2 \sim c_s^2$$

To keep the contributions at the same order of magnitude, we can take for example

$$\beta = \gamma\alpha, \quad c_s = \frac{1}{\sqrt{\alpha}}$$