An Eulerian hyperbolic model for heat transfer derived via Hamilton's principle

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Diffusion equations

- Many phenomena in nature are described by diffusion-type equations

1. Fick's second law for particle concentration

$$\frac{\partial c}{\partial t} = \mathrm{div}\left(D\nabla c\right)$$

2. Fourier's law for heat conduction

$$\frac{\partial \theta}{\partial t} = \mathrm{div}\left(K\nabla \theta\right)$$

3. etc ...

Very "simple" structure, compares well with experimental observations.

Heat conduction in an inviscid compressible flow

Generally described by Euler equation + Fourier's law of heat conduction

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1a}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + p(\rho, \eta)\mathbf{I}) = 0, \tag{1b}$$

$$\frac{\partial E}{\partial t} + \operatorname{div}(E\mathbf{u} + p(\rho, \eta)\mathbf{u} - K\nabla\theta(\rho, \eta)) = 0. \tag{1c}$$

System describes conservation of mass, momentum and total energy.

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System describes conservation of mass, momentum and total energy.
Entropy satisfies Clausius-Duhem inequality

$$\frac{\partial \rho \eta}{\partial t} + \mathrm{div}\left(\rho \eta \mathbf{u} - \frac{K}{\theta}\nabla\theta\right) = \frac{K}{\theta^2}\left|\left|\nabla\theta\right|\right|^2 \geq 0.$$

Objective

We would like to provide first-order hyperbolic alternative to the Euler-Fourier system

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{2a}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + p(\rho, \eta)\mathbf{I}) = 0, \tag{2b}$$

$$\frac{\partial E}{\partial t} + \operatorname{div}(E\mathbf{u} + p(\rho, \eta)\mathbf{u} - K\nabla\theta(\rho, \eta)) = 0. \tag{2c}$$

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   information must not travel faster than light speed in vacuum. (Trivially violated by Laplace operator)
2. Symmetric hyperbolic equations are locally well-posed.
3. Obtain an alternative description of known phenomena.
4. Chance it provides much easier/faster numerical simulations (Very often the case, not always).

Plan of presentation

Objective properties

We want to obtain a model that satisfies the following properties

1. Can be derived from a variational principle
2. First-order hyperbolic system
3. Can be cast into a Friedrichs symmetric form
4. Total Energy is conserved
5. Compatible with the second law of thermodynamics
6. Gallilean invariant

Cattaneo's model

Well-known hyperbolic relaxation of heat equation (1948)

$$\frac{\partial \theta}{\partial t} + \operatorname{div}(\mathbf{q}) = 0$$
$$\tau \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -K \nabla \theta$$

- No underlying variational principle.
- Not Gallilean invariant (fixable)
- Generally considered out of the scope of fluid dynamics.
- when coupled with compressible Euler equations
  1. Not hyperbolic in multi-D.
  2. Does not satisfy Clausius-Duhem inequality.

Cattaneo's model

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + \left(\rho u^2 + p\right)_x = 0$$

$$\left(\rho\left(\varepsilon + \frac{1}{2}u^2\right)\right)_t + \left(\rho u\left(\varepsilon + \frac{1}{2}u^2\right)\right)_x = -(pu)_x - q_x$$

$$\tau q_t + \tau u q_x + q = -\kappa\theta_x$$

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\end{aligned}
$$

The Gibbs identity implies that

$$
\rho\theta\frac{d\eta}{dt} = -q_x
$$

which can be cast in conservative form as

$$
(\rho\eta)_t + \left(\rho\eta u + \frac{q}{\theta}\right)_x = -\frac{q\theta_x}{\theta^2} >? \ 0
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not necessarily satisfied because $q \neq -K\theta_x \quad \forall\tau > 0$.

About Euler-Lagrange equations

Given a Lagrangian, you can derive the Euler-Lagrange equation

$$\mathcal{L}(q, \dot{q}, \nabla q) \quad \Longrightarrow \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) + \mathrm{div}\left(\frac{\partial \mathcal{L}}{\partial \nabla q}\right) = \frac{\partial \mathcal{L}}{\partial q}$$

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Things are already more complicated for Euler equations

$$\mathcal{L}(\rho, \mathbf{u}) = \int_{\Omega_t} \left(\frac{1}{2}\rho \left|\left|\mathbf{u}\right|\right|^2 - \rho\varepsilon(\rho, \eta)\right) \, d\Omega,$$

$$\delta\rho = -\mathrm{div}\left(\rho \boldsymbol{\delta x}\right), \quad \delta\mathbf{u} = \frac{\partial \boldsymbol{\delta x}}{\partial t} + \frac{\partial \boldsymbol{\delta x}}{\partial \mathbf{x}}\mathbf{u} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\boldsymbol{\delta x}$$

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After a bit of calculus $\quad \Rightarrow \quad \dfrac{\partial \rho\mathbf{u}}{\partial t} + \mathrm{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + \rho^2\dfrac{\partial \varepsilon}{\partial \rho}\mathbf{I}\right) = 0$

Euler equations for compressible fluids

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \mathbf{u}) = 0, \quad \text{(mass)}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \mathrm{div}\,(\rho \mathbf{u} \otimes \mathbf{u} + p(\rho, \eta)\mathbf{I}) = 0, \quad \text{(momentum)}$$

$$\frac{\partial \rho \eta}{\partial t} + \mathrm{div}\,(\rho \eta \mathbf{u}) = 0. \quad \text{(entropy)}$$

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$$\frac{\partial \rho \eta}{\partial t} + \operatorname{div}(\rho \eta \mathbf{u}) = 0. \quad \text{(entropy)}$$

Summing up these equations yields the energy conservation equation

$$\frac{\partial E}{\partial t} + \operatorname{div}(E\mathbf{u} + p(\rho, \eta)\mathbf{u}) = 0. \quad \text{(Energy)}$$

Thermal displacement (Green-Naghdi 1991)

In this paper :

[1] Green, A. E., & Naghdi, P. (1991). A re-examination of the basic postulates of thermomechanics. Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences, 432(1885), 171-194.

The authors introduce an independent auxiliary potential $\phi(\mathbf{x}, t)$ as a thermal analogue of the kinematic variables such that

$$\dot{\phi}(\mathbf{x}, t) = -\theta(\mathbf{x}, t)$$

One can then write the Lagrangian

$$\mathcal{L}(\rho, \mathbf{u}, \dot{\phi}) = \int_{\Omega} \left(\frac{1}{2} \rho \left|\left|\mathbf{u}\right|\right|^2 - \rho \varepsilon^{\star}(\rho, \dot{\phi}) \right) \, d\Omega,$$

where

$$\varepsilon(\rho, \eta) = \varepsilon^{\star}(\rho, \dot{\phi}) - \eta \dot{\phi}, \quad \text{with} \quad \eta = \frac{\partial \varepsilon^{\star}}{\partial \dot{\phi}}.$$

Entropy as an Euler-Lagrange equation

Given the Lagrangian

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One obtains

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \operatorname{div} \left(\rho \mathbf{u} \otimes \mathbf{u} + \rho^2 \frac{\partial \varepsilon^\star}{\partial \rho} \mathbf{I} \right) = 0, \quad \text{(Euler-Lagrange for } \delta \mathbf{x} \text{)}$$

$$\frac{\partial}{\partial t} \left(\rho \frac{\partial \varepsilon^\star}{\partial \dot{\phi}} \right) + \operatorname{div} \left(\rho \frac{\partial \varepsilon^\star}{\partial \dot{\phi}} \mathbf{u} \right) = 0, \quad \text{(Euler-Lagrange for } \delta \phi \text{)}$$

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$$\frac{\partial}{\partial t}(\rho \eta) + \operatorname{div}(\rho \eta \mathbf{u}) = 0, \quad \text{(Euler-Lagrange for } \delta \phi\text{)}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0 \quad \text{(Constraint)}$$

- A similar idea was also used in Lagrangian coordinates in *Peshkov, Pavelka, Grmela and Romenski (2018)*.

Extension of Green-Naghdi's philosophy

Consider the Lagrangian

$$\mathcal{L}(\rho, \mathbf{u}, \nabla\phi, \dot{\phi}) = \int_{\Omega} \left(\frac{1}{2}\rho \left|\left|\mathbf{u}\right|\right|^2 - \rho\varepsilon^{\star}(\rho, \dot{\phi}) - \frac{1}{2}\alpha(\rho) \left|\left|\nabla\phi\right|\right|^2 \right) \, d\Omega,$$

where the function $\alpha(\rho)$ is an arbitrary positive function of density.

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&\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = 0,\\
&\frac{\partial\rho\mathbf{u}}{\partial t} + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u} + P\,\mathbf{I} + \alpha(\rho)\ \nabla\phi\otimes\nabla\phi) = 0,\\
&\frac{\partial\rho\eta}{\partial t} + \operatorname{div}(\rho\eta\mathbf{u} + \alpha(\rho)\nabla\phi) = 0,
\end{aligned}$$

where $P = \rho^2\dfrac{\partial\varepsilon^\star}{\partial\rho} + \dfrac{1}{2}(\rho\alpha'(\rho) - \alpha(\rho)) \left|\left|\nabla\phi\right|\right|^2$

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- Problem : PDE is of second order and depends on $\nabla\phi$.

Solution: First-order reduction

Recall that

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = -\theta(\rho, \eta)$$

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$$\frac{\partial \nabla \phi}{\partial t} + \nabla \left(\mathbf{u} \cdot \nabla \phi + \theta(\rho, \eta) \right) = 0$$

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$$\frac{\partial \nabla \phi}{\partial t} + \nabla \left(\mathbf{u} \cdot \nabla \phi + \theta(\rho, \eta) \right) = 0$$

Let us introduce $\mathbf{j} = \nabla \phi$ as an independent variable. Then $\mathbf{j}$ satisfies

$$\frac{\partial \mathbf{j}}{\partial t} + \nabla \left(\mathbf{u} \cdot \mathbf{j} + \theta(\rho, \eta) \right) = 0$$

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$$\nabla\left(\frac{\partial\phi}{\partial t}\right)+\nabla(\mathbf{u}\cdot\nabla\phi)=-\nabla(\theta(\rho,\eta))$$

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$$\frac{\partial\mathbf{j}}{\partial t}+\nabla\left(\mathbf{u}\cdot\mathbf{j}+\theta(\rho,\eta)\right)=0$$

Note that since $\mathbf{j}=\nabla\phi$ then $\mathbf{j}$ satisfies

$$\nabla\times\mathbf{j}=0.$$

Dissipationless system of equations

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \\
&\frac{\partial \rho \mathbf{u}}{\partial t} + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + \Pi) = 0, \quad \Pi = P(\rho, \eta, \mathbf{j})\, \mathbf{I} + \alpha(\rho)\, \mathbf{j} \otimes \mathbf{j} \\
&\frac{\partial \mathbf{j}}{\partial t} + \nabla(\mathbf{j} \cdot \mathbf{u} + \theta(\rho, \eta)) + \left(\frac{\partial \mathbf{j}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{j}}{\partial \mathbf{x}} \right)^T \right) \mathbf{u} = 0, \\
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&\frac{\partial \rho \eta}{\partial t} + \mathrm{div}\left(\rho \eta \mathbf{u} + \alpha(\rho)\mathbf{j}\right) = 0.
\end{aligned}
$$

Total energy conservation is obtained as a consequence

$$
\frac{\partial E}{\partial t} + \mathrm{div}\left(E\mathbf{u} + \Pi \mathbf{u} + \mathbf{q}\right) = 0, \quad \mathbf{q} = \alpha(\rho)\, \theta(\rho, \eta)\, \mathbf{j}
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$$

Additional term in the energy conservation *should* be the heat flux.

Rayleigh dissipation function

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \\
&\frac{\partial \rho \mathbf{u}}{\partial t} + \operatorname{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + P(\rho, \eta, \mathbf{j})\, \mathbf{I} + \alpha(\rho)\, \mathbf{j} \otimes \mathbf{j}\right) = 0, \\
&\frac{\partial \mathbf{j}}{\partial t} + \nabla\left(\mathbf{j} \cdot \mathbf{u} + \theta(\rho, \eta)\right) + \left(\frac{\partial \mathbf{j}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{j}}{\partial \mathbf{x}}\right)^T\right) \mathbf{u} = -\frac{\partial \mathcal{R}}{\partial \mathbf{j}}, \\
&\frac{\partial \rho \eta}{\partial t} + \operatorname{div}\left(\rho \eta \mathbf{u} + \alpha(\rho) \mathbf{j}\right) = \frac{\alpha(\rho)}{\theta(\rho, \eta)} \frac{\partial \mathcal{R}}{\partial \mathbf{j}} \cdot \mathbf{j}.
\end{aligned}
$$

Here $\mathcal{R}$ is the *Rayleigh dissipation* function and which we take in the simplest form as

$$
\mathcal{R} = \frac{1}{2\tau} \|\mathbf{j}\|^2, \qquad \frac{\partial \mathcal{R}}{\partial \mathbf{j}} = \frac{1}{\tau} \mathbf{j}
$$

Asymptotic Analysis and compatibility with Fourier's law

we can expand the variables in power series of τ

$$\rho = \rho_0 + O(\tau), \quad \mathbf{u} = \mathbf{u}_0 + O(\tau), \quad \eta = \eta_0 + O(\tau), \quad \mathbf{j} = \mathbf{j}_0 + \tau \mathbf{j}_1 + o(\tau),$$

and we focus on the $\mathbf{j}$ equation

$$\tau \left(\frac{\partial \mathbf{j}_0}{\partial t} + \frac{\partial \mathbf{j}_0}{\partial \mathbf{x}} \mathbf{u}_0 + \left(\frac{\partial \mathbf{u}_0}{\partial \mathbf{x}} \right)^T \mathbf{j}_0 + \nabla \theta(\rho_0, \eta_0) \right) = -(\mathbf{j}_0 + \tau \mathbf{j}_1) + o(\tau).$$

to obtain

$$\mathbf{j}_0 = 0, \quad \mathbf{j}_1 = -\nabla \theta(\rho_0, \eta_0), \quad \Longrightarrow \boxed{\quad \mathbf{j} = -\tau \; \nabla \theta(\rho_0, \eta_0) + o(\tau).}$$

Under these considerations, the heat flux vector expresses as

$$\mathbf{q} = -\tau \alpha(\rho_0) \, \theta(\rho_0, \eta_0) \, \nabla \theta(\rho_0, \eta_0).$$

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Under these considerations, the heat flux vector expresses as

$$\mathbf{q} = -\tau \alpha(\rho_0) \, \theta(\rho_0, \eta_0) \, \nabla \theta(\rho_0, \eta_0).$$

compatible with Fourier's law if

$$\tau = \frac{K}{\alpha(\rho_0) \ \theta(\rho_0, \eta_0)}$$

Energy convexity

Total energy is given by

$$E(\rho, \mathbf{m}, s, \mathbf{j}) = \frac{1}{2\rho}\left|\left|\mathbf{m}\right|\right|^2 + \rho\varepsilon(\rho, s/\rho) + \frac{1}{2}\alpha(\rho)\left|\left|\mathbf{j}\right|\right|^2, \quad \mathbf{m} = \rho\mathbf{u}, s = \rho\eta$$

Sufficient criterion for energy convexity

$$\text{if } \frac{\partial^2}{\partial\rho^2}\left(\frac{1}{\alpha(\rho)}\right) \leq 0, \quad \text{for } \rho > 0.$$

then E i s also a convex function of $\mathbf{Q}$.

We choose a simple function fitting this criterion

$$\alpha(\rho) = \frac{\varkappa^2}{\rho}, \quad \varkappa = cst.$$

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(Another possibility is $\alpha(\rho) = cst$, taken in *Peshkov et.al. (2018)*)

Hyperbolicity

system can be cast into quasilinear form

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A}(\mathbf{V})\frac{\partial \mathbf{V}}{\partial \mathbf{x}} = 0$$

where $\mathbf{A}$ admits 8 eigenvalues whose expressions are given by

$$\begin{cases} \chi_1 = u_1 - \sqrt{Z_1 + Z_2}, \\ \chi_2 = u_1 - \sqrt{Z_1 - Z_2}, \\ \chi_{3-6} = u_1, \\ \chi_7 = u_1 + \sqrt{Z_1 - Z_2}, \\ \chi_8 = u_1 + \sqrt{Z_1 + Z_2} \end{cases} \quad \text{where} \begin{cases} Z_1 = \dfrac{1}{2}\left(a_p^2 + a_T^2 + a_j^2\right), \\ Z_2 = \sqrt{a_{pT}^4 + \dfrac{1}{4}\left(a_p^2 - a_T^2\right)^2}, \\ a_p^2 = \dfrac{\partial p}{\partial \rho}, \quad a_T^2 = \dfrac{\varkappa^2}{\rho^2}\dfrac{\partial \theta}{\partial \eta}, \\ a_{pT}^4 = \dfrac{\varkappa^2}{\rho^2}\dfrac{\partial p}{\partial \eta}\dfrac{\partial \theta}{\partial \rho}, \quad a_j^2 = \dfrac{2\varkappa^2}{\rho^2}\left(j_2^2 + j_3^2\right). \end{cases}$$

Limiting behavior in 1D

Eigenvalues

$$\lambda_{1,4} = u_1 \pm \sqrt{Z_1 + Z_2}, \qquad \lambda_{2,3} = u_1 \pm \sqrt{Z_1 - Z_2}.$$

In the asymptotic limit $\varkappa \to \infty$ we have

$$\lim_{\varkappa \to \infty} \chi_{1,8} = \pm\infty, \quad \lim_{\varkappa \to \infty} \lambda_{2,3} = u_1 \pm a_\theta$$

with a_θ being the isothermal sound speed given by

$$a_\theta = \sqrt{\frac{\partial \tilde{p}(\rho, \theta)}{\partial \rho}} = \sqrt{\frac{\partial p(\rho, \eta)}{\partial \rho} - \frac{\partial p(\rho, \eta)}{\partial \eta} \frac{\partial \theta}{\partial \rho} \Big/ \frac{\partial \theta}{\partial \eta}}$$

Dispersion relation comparison

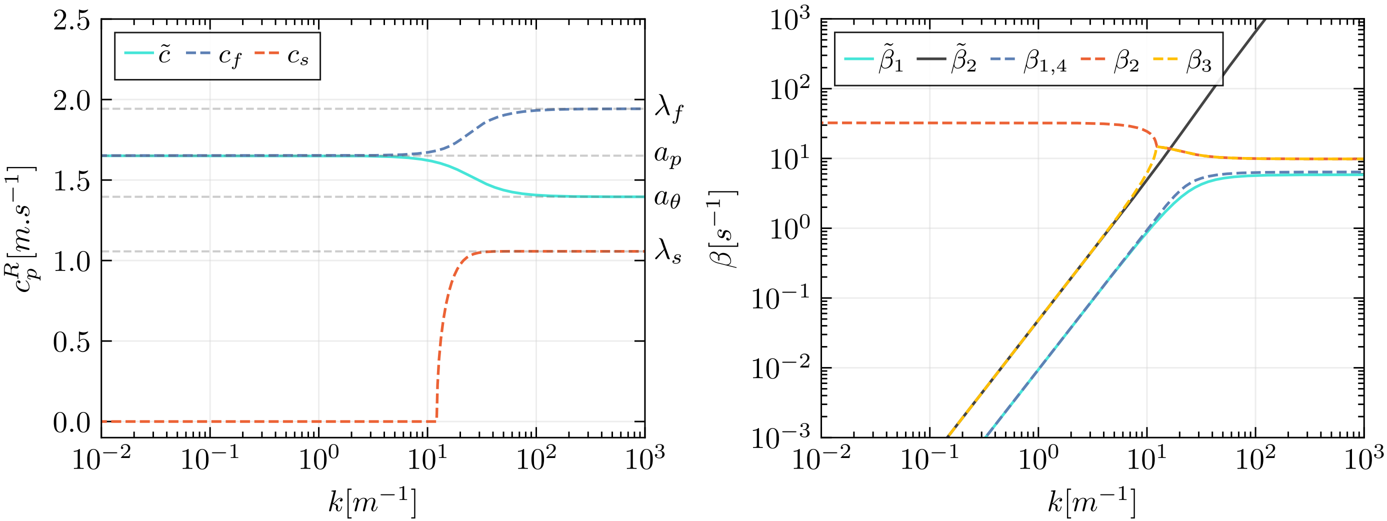

Figure 1: log-linear plot of the norms of the real part of the phase velocities (Left) and log-log plot of the attenuation factors (Right) for both hyperbolic system (Dashed lines) and original Euler-Fourier system (Solid lines).

$$a_p = \sqrt{\frac{\partial p}{\partial \rho}}, \qquad a_\theta = \sqrt{\frac{\partial p}{\partial \rho} - \frac{\partial p}{\partial \eta}\frac{\partial \theta}{\partial \rho} \Big/ \frac{\partial \theta}{\partial \eta}}.$$

1D-study: Eigenfields

In one dimension of space, we can write the system as

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0,\\
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial \eta}\frac{\partial \eta}{\partial x} = 0,\\
&\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + \frac{\varkappa^2}{\rho^2}\frac{\partial j}{\partial x} - \frac{\varkappa^2}{\rho^3}j\frac{\partial \rho}{\partial x} = 0,\\
&\frac{\partial j}{\partial t} + j\frac{\partial u}{\partial x} + u\frac{\partial j}{\partial x} + \frac{\partial \theta}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial x} = 0.
\end{aligned}$$

The eigenvalues are given by

$$\begin{cases}
\lambda_1 = u - \sqrt{Y_1 + Y_2},\\
\lambda_2 = u - \sqrt{Y_1 - Y_2},\\
\lambda_3 = u + \sqrt{Y_1 - Y_2},\\
\lambda_4 = u + \sqrt{Y_1 + Y_2},
\end{cases}
\quad \text{where} \quad
\begin{cases}
Y_1 = \frac{1}{2}\left(a_p^2 + a_T^2\right),\\
Y_2 = \sqrt{a_{pT}^4 + Y_3^2},\\
Y_3 = \frac{1}{2}\left(a_p^2 - a_T^2\right).
\end{cases}$$

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- System admits full basis of eigenvectors.

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Nature of the eigenfields (polytropic gas equation of state):

- System admits full basis of eigenvectors.
- Eigenfields associated to $\lambda_{1,4}$ are genuinely non-linear.
- Eigenfields associated to $\lambda_{2,3}$ are neither genuinely non-linear, neither linearly degenerate.

Rankine-Hugoniot conditions

In one dimension of space the RH conditions write

$$
\begin{aligned}
[\mathcal{M}] &= 0,\\
\left[p + \frac{\mathcal{M}^2}{\rho}\right] &= 0,\\
\left[\mathcal{M}\left(\frac{\mathcal{M}^2}{2\rho^2} + \varepsilon + \frac{p}{\rho} + \frac{1}{2}\frac{\varkappa^2}{\rho^2} j^2\right) + \frac{\varkappa^2}{\rho}\theta\, j\right] &= 0,\\
\left[\mathcal{M}\frac{j}{\rho} + \theta\right] &= 0,
\end{aligned}
$$

where we have defined the mass flux across the discontinuity front by $\mathcal{M} = \rho(u - \mathcal{D})$ and $\mathcal{D}$ is the discontinuity speed.

Non-existence of contact discontinuity

On contact discontinuities, that is for $\mathcal{M} = 0$, one obtains by direct substitution

$$[p] = 0, \quad \left[\varkappa^2 \frac{j}{\rho} \theta\right] = 0, \quad [\theta] = 0.$$

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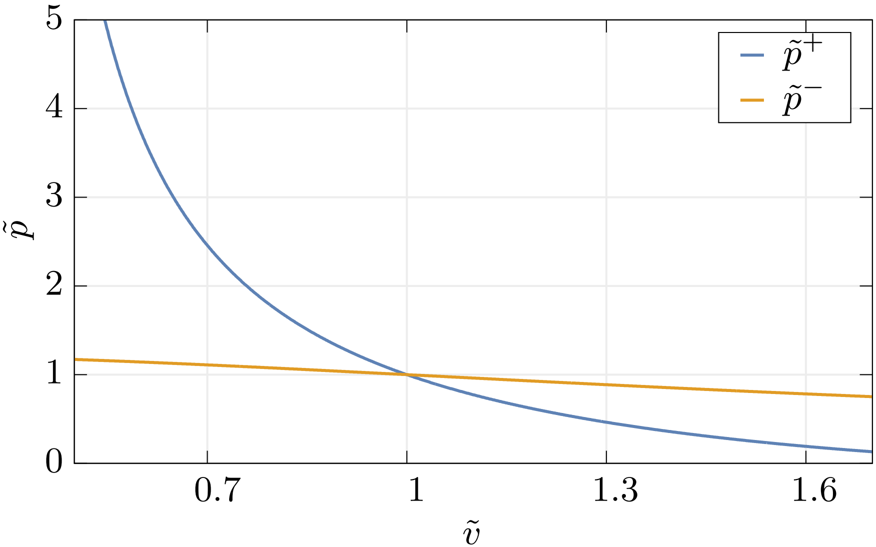
Since p and θ are continuous across the discontinuity, the density will be as well. Thus:

$$[\rho] = 0, \quad [u] = 0, \quad [\eta] = 0, \quad [j] = 0,$$

Therefore solution is continuous: no contact discontinuities are admissible in this case.

Hugoniot Locus (polytropic gas equation of state)



Study of the Hugoniot curves shows interesting possible solutions:

- Expansion shocks,
- Compression fans,
- Compound shocks.

Compound shocks

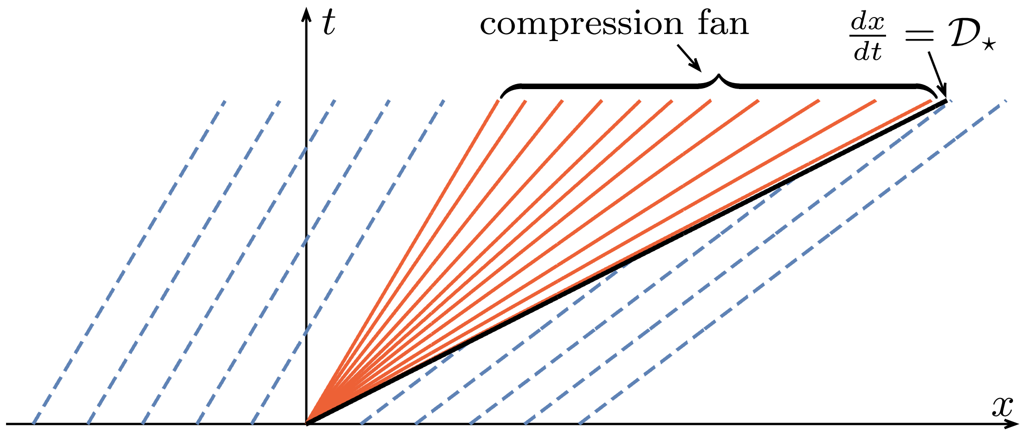


Figure 2: Schematic representation of the wave pattern in the $x - t$ plane, for a compound shock splitting solution. The shock propagates to the right, followed by a right facing compression fan.

Recovery of Fourier law: Shock tube problem

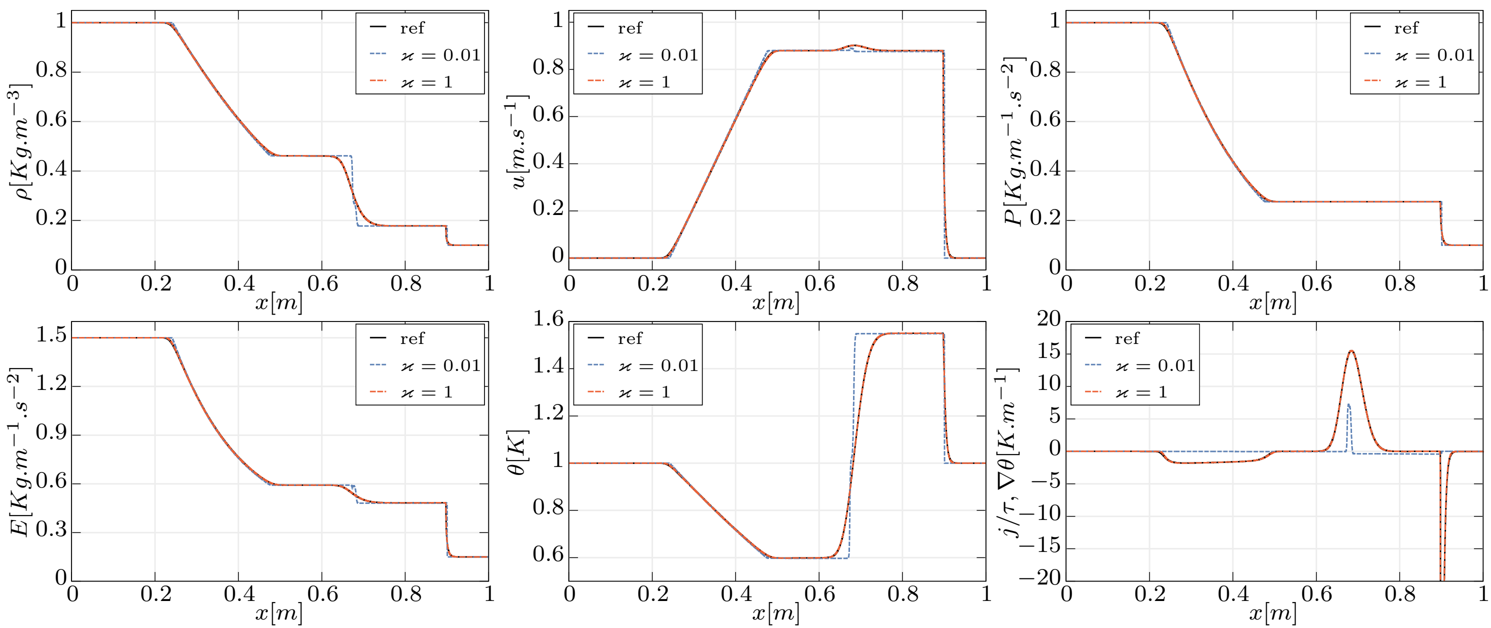


Figure 3: Shock tube with heat conduction. The solution is given at final time $t = 0.2$. Parameters: $\mathrm{CFL} = 0.9$, $\gamma = 5/3$, $c_V = 3/2$, $K = 10^{-3}$. Relaxation time is taken as $\tau = \frac{K}{\alpha(\rho_0)\,\theta(\rho_0,\eta_0)}$

Expansion shock solution

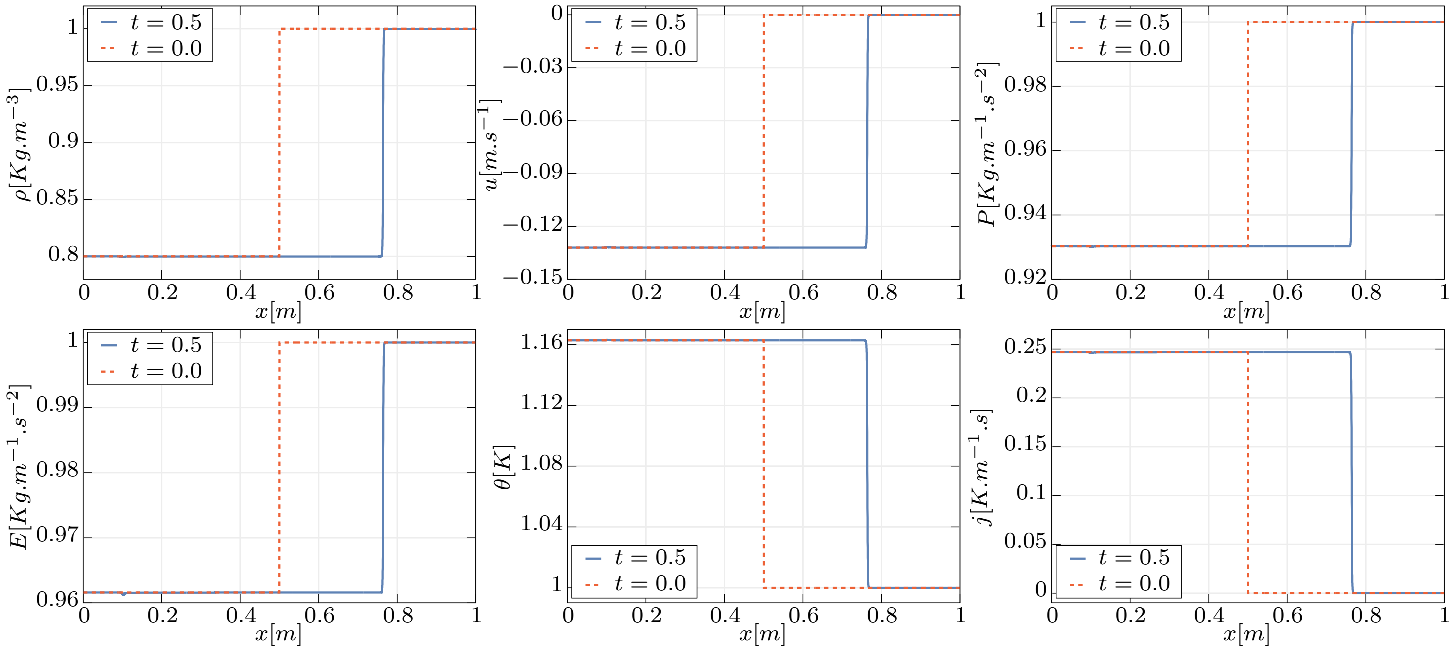


Figure 4: Numerical result for an expansion shock solution on the computational domain $[0, 1]$, discretized over $N = 10000$ cells displayed at final time $t = 0.5$. Parameters: $\mathrm{CFL} = 0.9$, $\gamma = 2$, $c_V = 1$, $\varkappa = 0.8$.

Compression fan

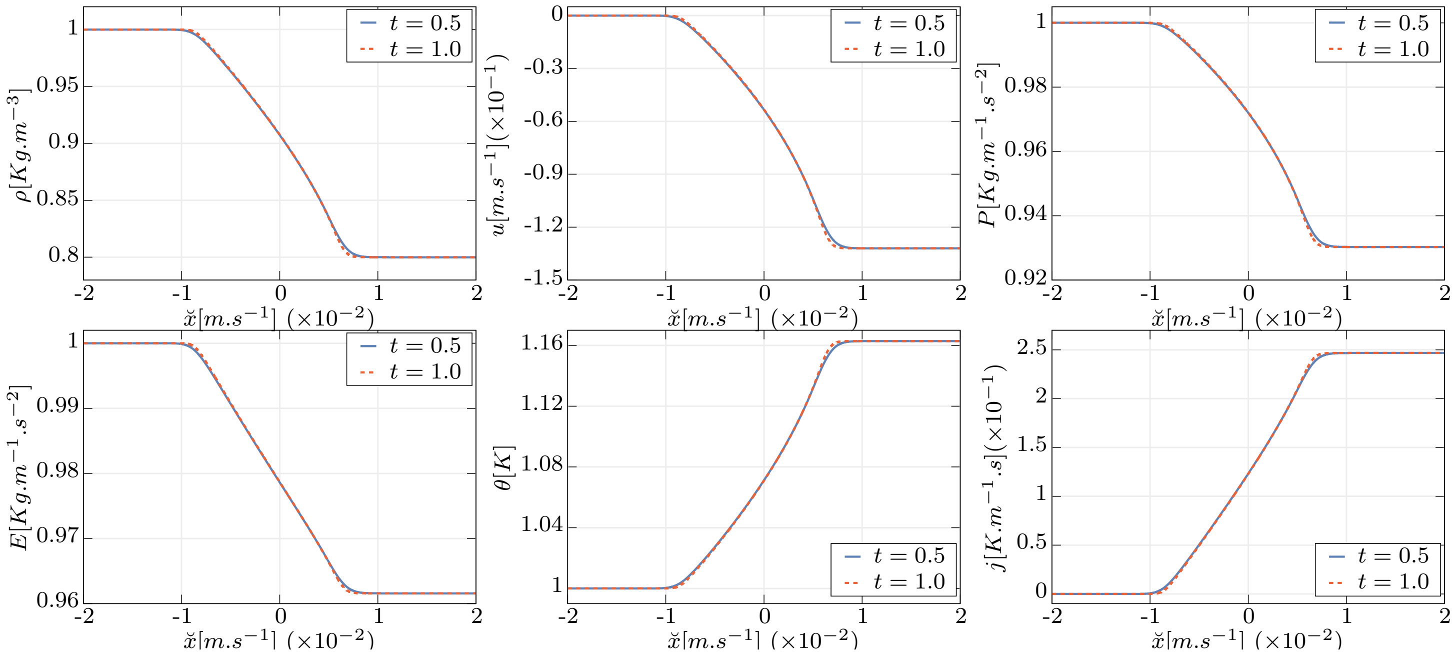


Figure 5: Numerical result for a compression fan solution. Parameters: $\mathrm{CFL} = 0.9$, $\gamma = 2$, $c_V = 1$, $\varkappa = 0.8$

Compound shock solution

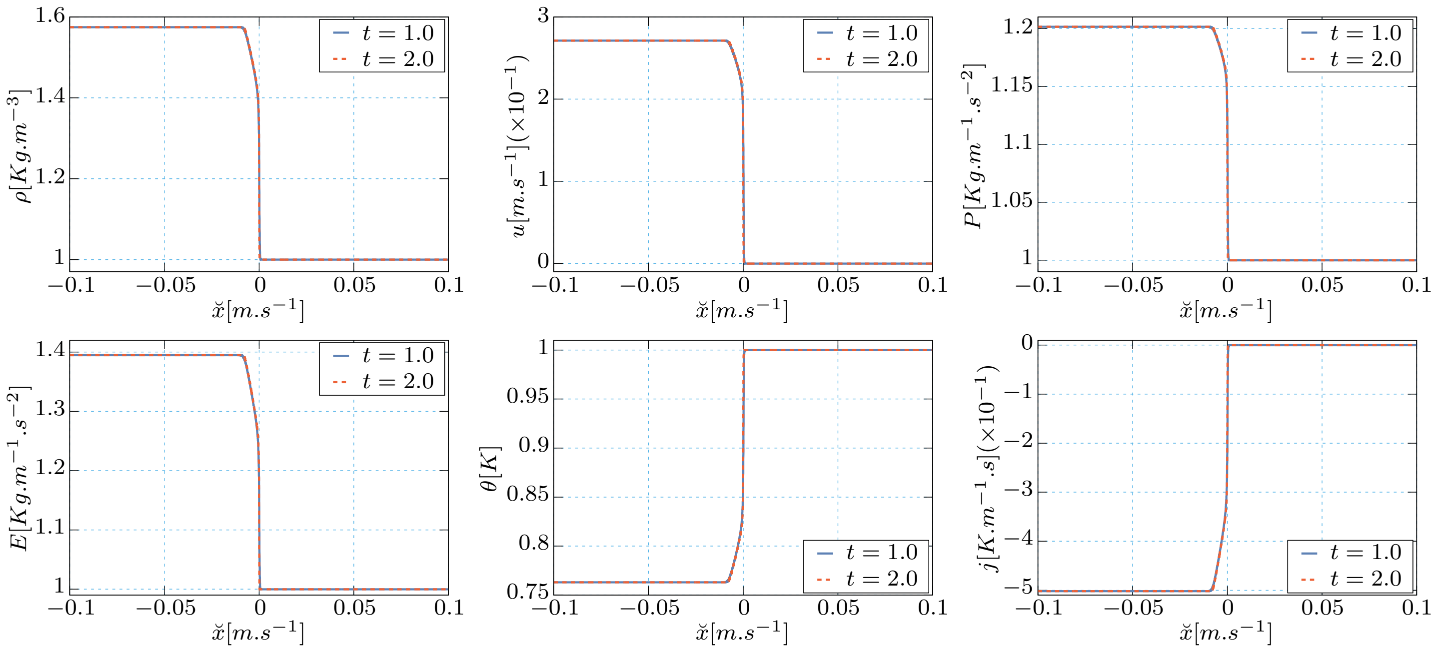


Figure 6: Compound shock plotted as a function of the self-similar coordinate $\breve{x} = (x - \mathcal{D}_\star t)/t$. CFL $= 0.9$, $\gamma = 2$, $c_V = 1$, $\varkappa = 1.3$.

Conclusion and Perspectives

- Heat conduction can be modeled by hyperbolic equations derived from variational principles.
- Entropy equation can be derived as an Euler-Lagrange equation.

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Perspectives

- Multi-D simulations (accounting for curl-involutions, etc)
- Rigorous Justification of the relaxation limit
- Further optimization at the numerical level (semi-implicit discretization, etc)
- Further study of the Riemann problem.

Thank you for your attention !

[1] Dhaouadi, Firas, and Sergey Gavrilyuk. "An Eulerian hyperbolic model for heat transfer derived via Hamilton's principle: analytical and numerical study." Proceedings of the Royal Society A 480.2283 (2024): 20230440.

And references therein.

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